Nearby Lagrangian Conjecture and Homotopy Spheres

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Joint work in progress with D. Auroux, L. Grama, L. Katzarkov, and L. Sperança

Recap

- Q: smooth manifold. T^*Q is canonically an exact symplectic manifold (T^*Q, ω) with primitive 1-form $\omega = \mathrm{d}\lambda$

$$\lambda_{(p,q)}(v) := p(\pi_* v)$$

$q \in Q,\ p \in T^*_q Q,\ \pi : TQ \to Q,\ v \in T_{(p,q)} T^*Q.$

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- **Question**(a). (Arno'ld, Eliashberg, ...): if Q is closed then is (T^*Q, ω) a complete diffeomorphism invariant of Q?
- If Q is open, the h-principle gives an affirmative answer (despite the existence of uncountably many non-pairwise diffeomorphic $\mathbb{R}^4$).

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Any Lagrangian in the Euclidean space is a Lagrangian in any symplectic manifold of the same dimension.

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- **Example**. In $T^*\mathrm{S}^1 \cong \mathrm{S}^1 \times \mathbb{R}$, any circle drawn in the cylinder is a Lagrangian, but an *exact* Lagrangian circle must have the same area above the zero section as that below the zero section.
- **Definition**. Embedded closed exact Lagrangians in (T^*Q, ω) are termed *nearby Lagrangians*.

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(Weaker form of) The Nearby Lagrangian Conjecture

- **Question**.(a') Is any nearby Lagragian $L \hookrightarrow (T^*Q, \omega)$ diffeomorphic to Q? It is homotopically equivalent, however! (Abouzaid, Kragh, Fukaya-Seidel-Smith, Nadler, Zaslow, ...).
- Abouzaid: Every nearby Lagrangian in $T^*\mathrm{S}^{4k+1}$ **bounds** a compact parallelisable manifold.

The bounding manifold (Abouzaid)

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- The moduli space arising from a 1-parameter family of perturbations of the Cauchy–Riemann equation is parallelizable (assuming suitable transversality).
- By Gromov's argument, this moduli space cannot be compact: non-compactness comes from either
  - sphere bubbling, or
  - disc bubbling.

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Theorem. (Ekholm–Smith) *In even dimension n, a homotopy sphere Σ^n admits an exact Lagrangian immersion $\Sigma^n \looparrowright \mathbb{C}^n$ with exactly one transverse double point (and no other self-intersections) if and only if Σ^n is diffeomorphic to the standard sphere S^n.*

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- Hitchin, Lawson, Yau: There are 10-dimensional exotic spheres that do not bound spin manifolds and do not admit positive scalar curvature metrics.

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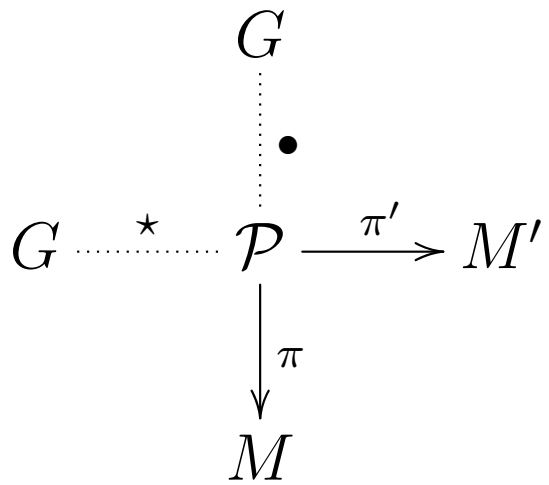
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Actually: $(T^*M, G) \sim (T^*M', G)$ as Morita equivalence for foliations. Lawson-Yau, Cavenaghi-do Ó-Sperança: M has positive scalar curvature if, and only if M' does (for some metric).

Kervaire spheres

The sphere S^{4m+3} can be described as the union of two closed disks D^{4m+3}, glued along their common boundary S^{4m+2} via a map

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Decomposition of the Sphere S^{4m+3}

Equivariance Condition on the Boundary

Assume that: the boundary sphere $\mathrm{S}^{4m+2} \cong \partial \mathrm{D}^{4m+3}$ carries a G-action compatible with ϕ

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This equivariance allows the definition of the associated *autophagic* diffeomorphism:

$$\widehat{\phi} : \mathrm{S}^{4m+2} \to \mathrm{S}^{4m+2}, \quad \widehat{\phi}(u) := \phi(u) \cdot u.$$

Gluing Two Disks via $\widehat{\phi}$

Using $\widehat{\phi}$, we glue two copies of D^{4m+3} along their boundary to reconstruct S^{4m+3}:

$$\mathrm{S}^{4m+3} \cong \mathrm{D}^{4m+3} \cup_{\widehat{\phi}} \mathrm{D}^{4m+3}. \tag{1}$$

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Kervaire spheres Σ^n are the boundaries of the plumbing of two copies of the disc bundle associated to the tangent bundle of S^n, where $n = 2m + 1$. The Kervaire sphere is known to be exotic for all but finitely many known values of m (Hill–Hopkins–Ravenel, Wang–Xu).

Principal $\mathrm{O}(n)$-Bundle over S^n

Consider the bundle $\mathrm{pr}_n : \mathrm{O}(n+1) \to \mathrm{S}^n$ with right principal action:

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- Fixes first column $v_0 = Re_0$
- Rotates remaining n vectors

Quotient Identification

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Projection map: $(\mathrm{pr}_n)' : \mathrm{O}(n+1) \to \mathrm{S}^n$

Bundle Reduction to $\mathrm{U}(m+1)$

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Clutching Construction

Define transition function on equator S^{2m}:

$$\tau_m^{\mathbb{C}} \begin{pmatrix} y \\ z \end{pmatrix} := \left(\mathrm{id} - \frac{z}{|z|}(1 + e^{\pi y}) \frac{\bar{z}^t}{|z|} \right)$$

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Thus: $\mathrm{U}(m+1) \cong \mathrm{S}^{2m+1} \times_{\tau_m^{\mathbb{C}}} \mathrm{U}(m)$

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The $(4m+1)$-Kervaire sphere is:

$$\Sigma^{4m+1} := (J\tau_m^{\mathbb{C}})^{-1}(\mathrm{D}_+^{2m+1}) \cup_{\widehat{\tau_m^{\mathbb{C}} \circ J\tau_m^{\mathbb{C}}}} (J\tau_m^{\mathbb{C}})^{-1}(\mathrm{D}_-^{2m+1})$$

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Gluing map:

$$\widehat{\tau_m^{\mathbb{C}} \circ J\tau_m^{\mathbb{C}}}(x_1, x_2) := \left[\tau_m^{\mathbb{C}} \circ J\tau_m^{\mathbb{C}}(x_1, x_2)\right] \cdot (x_1, x_2)$$

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Yields Kervaire sphere in dimension $8m + 5$ with analogous maps.

Idea: how to build a pattern out of a stripped fabric?

Can one rip apart $(T\Sigma^{8m+5}, \omega)$ and rebuilt $(T\Sigma^{8m+5}, \widetilde{\omega})$ in a such way they are symplectomorphic and S^{8m+5} is a nearby Lagrangian?

Key lemma. There is an isotopy between $\tau_m^{\mathbb{H}}$ and its inverse $(\tau_m^{\mathbb{H}})^{-1}$. Consequently, $\widehat{(\tau_m^{\mathbb{H}} \circ J\tau_m^{\mathbb{H}})^2}$ is isotopic to the identity.

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Proof. Recall that every octonion is a real linear combination of the unit octonions $\{1, e_1, \ldots, e_7\}$ where $1 \in \mathbb{R}$. Consider the path $\alpha(s) := \cos(s) + e_4 \sin(s)$ with $s \in [0, \pi]$.

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$$ {}^{\alpha(s)}\tau_m^{\mathbb{H}} \begin{pmatrix} y \\ z \end{pmatrix} := \left(\mathrm{id} - \frac{z}{|z|}(1 + e^{\pi\alpha(s)y})\frac{\bar{z}^t}{|z|} \right) $$

Matrix Family $\phi_\alpha(u)$

Define the rotation matrix:

$$A_\alpha := \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

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- $\alpha = \pi/2$:

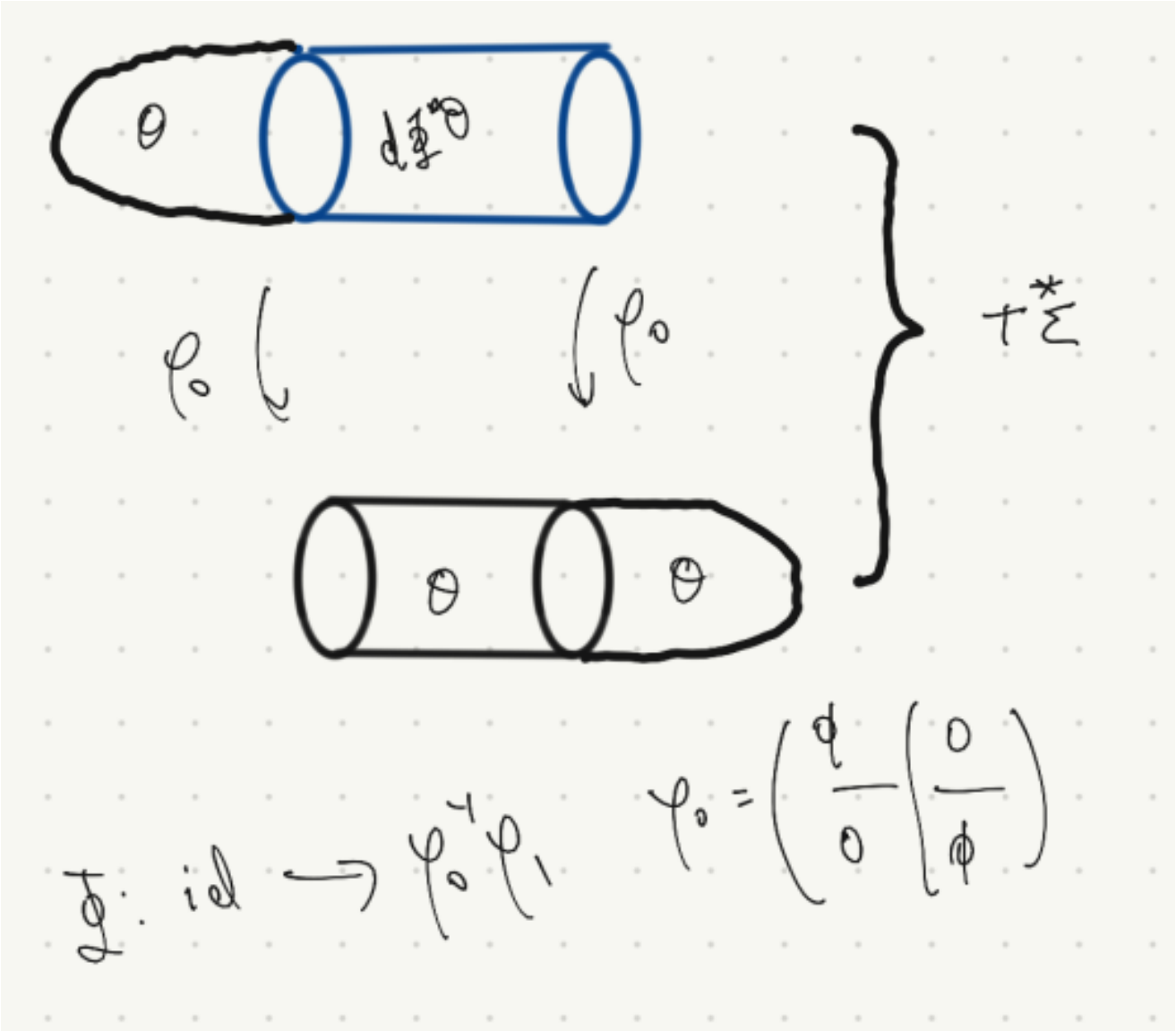
$$\phi_{\frac{\pi}{2}}(u) = \begin{pmatrix} \phi(u)^2 & 0 \\ 0 & I \end{pmatrix}$$

Mixing fibers and base

$$T_1\mathrm{S}^{8m+4} \to T_1\mathrm{S}^{8m+4}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto A_\alpha^T \begin{pmatrix} \phi(u) & 0 \\ 0 & I \end{pmatrix} A_\alpha \begin{pmatrix} I & 0 \\ 0 & \phi(u) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

A picture


φ_0
φ_0
$T^*\Sigma$
$\varphi_0 = \begin{pmatrix} \phi & 0 \\ 0 & \phi \end{pmatrix}$