G-equivariant atoms: present and future

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August 9, 2026

Plan

1. Setup: group actions and linearizability
2. A quick recall: atoms
3. G-equivariant atoms
4. Cubic fourfolds

The setting

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A G-invariant center need *not* be connected.

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- G-rational if $X \sim_G \mathbb{P}^n$ with *some* G-action.

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$$\text{(L)} \subseteq \text{(PL)} \subseteq \begin{cases} \text{(SL)} \\ G\text{-rational} \subseteq \text{rational} \end{cases}$$

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- Burnside symbols. Kontsevich–Pestun–Tschinkel, Kresch–Tschinkel; Cheltsov, Zhang, ...
  Assembled from the stratification by stabilizers.

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Output: the A-model F-(super)manifold B_X, a $\Bbbk$-analytic domain inside the analytification of $\operatorname{Spf} \mathrm{Nov}_X$, where

$$\mathrm{Nov}_X = \mathbb{Q}[[q_i, t_i]] \qquad \text{(big) Novikov ring.}$$

Two restrictions are not cosmetic:

- convergence: $B_{X,q} = \mathrm{val}^{-1}\big(\mathrm{Amp}(X)\big)$ inside $\big(\mathrm{NS}(X)_{\mathrm{tf}} \otimes \mathbb{G}_m\big)^{\mathrm{an}}$, and $|t_i| < 1$;
- maximality: the degree-2 coordinates are restricted to H^2_{trans} — otherwise the NS-part is counted twice (once by q, once by t).

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$$\mathrm{Eu}_b \;=\; c_1(T_X) \;+\; \left(\frac{\mathrm{Deg} - 2\cdot \mathrm{id}}{2}\right)(b),$$

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why would that have anything to do with birational geometry?

Iritani's blowup formula

Disjoint unions first. For $X \sqcup Z$ the Gromov–Witten potentials add variables, Thom–Sebastiani style:

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Now let $Z \subseteq X$ be smooth closed of codimension $r \geq 2$, with blowup $\pi\colon \widetilde{X} = \mathrm{Bl}_Z X \to X$ and exceptional divisor $E = \mathbb{P}(N_{Z/X})$. Put $X' = X \sqcup Z^{\sqcup(r-1)}$, so $B_{X'} = B_X \times (B_Z)^{\times(r-1)}$.

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Theorem 2 (Iritani, in the *A*-model *F*-manifold picture)

There are nonempty connected analytic domains $U_{\widetilde{X}} \subseteq B_{\widetilde{X}}$ and $U_{X'} \subseteq B_{X'}$ and an isomorphism of F-manifolds with Euler field over $\mathbb{k}$

$$\Psi\colon\ U_{\widetilde{X}} \xrightarrow{\ \sim\ } U_{X'}.$$

On rigid points Ψ restricts to a bijection whose tangent map is an isomorphism of unital $\mathbb{k}$-algebras intertwining the Euler operators, and Ψ is the unique analytic isomorphism inducing it.

Decomposition theorem

Definition 3

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Theorem 4 (KKPY)

For any $b \in B_X$ there is an analytic domain $U \ni b$ with

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Corollary 5 (KKPY)

$T_b B_X \cong \bigoplus_{i=1}^m \mathcal{A}_i(b)$, *each* $\mathcal{A}_i(b)$ *a unital algebra.*

So far: no Hodge theory. That is the next — and essential — ingredient.

Compatibility with Hodge structures

Let Hod be the pro-reductive group of $\mathbb{Z}/2$-folded polarizable pure Hodge structures:[2]

$$\mathrm{Rep}(\mathrm{Hod}) \;\cong\; \mathcal{HS} \,/\, \text{Tate twist}.$$

[2] Written MT in KKPY/CKK. The folding is a central ϵ of order 2 acting by $(-1)^{\mathrm{wt}}$.

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Crucially, Hod acts on B_X, and one must work on its fixed locus

$$B_X^{\mathrm{Hod}} \subset B_X \qquad \text{(the } \textit{Hodge locus}\text{)}.$$

Off B_X^{Hod} the operator $\boldsymbol{\kappa}_b$ does not commute with Hod, and the eigenspaces are not sub-Hodge structures.

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Theorem 6 (KKPY)

Let $U \subseteq B_X^{\mathrm{Hod}}$ be the open locus where the number of distinct eigenvalues of $\boldsymbol{\kappa}$ is maximal. For each rigid $b \in U$, every generalized eigenspace of $\boldsymbol{\kappa}_b$ carries a Hod*-module structure α and any rigid $b, b' \in U$ one has $\mathcal{A}_\alpha(b) \cong \mathcal{A}_\alpha(b')$ as* Hod*-modules. U may be chosen compatible with Iritani's blowup formula.*

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(Coarse) Hodge atoms

Theorem 7 (KKPY)

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Definition 8 (CKK after KKPY)

The (coarse) Hodge atom $\mathcal{A}_\alpha$ is the isomorphism class of the Hod-module carried by the sum of generalized eigenspaces for the eigenvalues parametrized by α.

Two key facts

Theorem 9 (KKPY, non-rationality criterion)

Let X be smooth projective of dimension $d \geq 2$. If X has a Hodge atom which does not occur in the atomic decomposition of any smooth projective variety of dimension $\leq d-2$, then X is not rational.

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The case we need

Σ a K3 $\Rightarrow$ $K_\Sigma \cong \mathcal{O}_\Sigma$ is nef $\Rightarrow$ one atom, namely all of $H^*(\Sigma)$:

$$\mathrm{rk} = 24, \qquad P(t) = t^2 + 22 + t^{-2}, \qquad \mathrm{lev} = 2, \quad \nu = 1.$$

Making atoms equivariant

Let a finite group G act on X. Then G acts on $H^*(X)$ and on B_X (on the Novikov directions through $G \curvearrowright \mathrm{NS}(X)$), and $G \times \mathrm{Hod}$ acts on $H^*(X, \mathbb{Q})$.

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Definition 12 (CKK, G-atoms)

The G-equivariant coarse atoms of (X, G) are the isomorphism classes of the $\overline{\mathbb{Q}}$-linear $G \times \mathrm{Hod}$-representations E^α attached to the local atoms.

Keeping track of G-atoms

Three numerical shadows of E^α:

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Read off P_α, using both ends:

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Example 13

- $\mathrm{lev} = 0 \iff \rho_\alpha = \dim \mathcal{A}_\alpha \iff \mathcal{A}_\alpha$ is of Tate type;
- H^4_{pr} of a cubic fourfold: $P = t^2 + 20 + t^{-2}$, so $\mathrm{lev} = 2$, $\nu = 1$;
- H^2 of a K3: $P = t^2 + 20 + t^{-2}$ — the same polynomial.

The non-equivariant theorem

For a smooth cubic fourfold $X \subset \mathbb{P}^5$: $b_4 = 23$, H^4_{pr} has rank 22, and $h^{3,1} = h^{1,3} = 1$, $h^{2,2}_{\mathrm{pr}} = 20$.

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Theorem 14 (Guéré)

If a smooth cubic fourfold X is rational, then there is a projective K3 surface Σ and an isomorphism of $\mathbb{Q}$-Hodge structures

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Mechanism. Track the atoms along a weak factorization $X \dashrightarrow \mathbb{P}^4$:

- X has an atom with $\mathrm{lev} = 2$, $\nu = 1$;
- $\mathbb{P}^4$ has $\mathrm{lev} = 0$ everywhere;
- lev and ν are additive, and centers of dimension ≤ 1 are of Tate type.

So some surface center must carry $\nu = 1$.

The atom of a cubic fourfold, I: the ambient part

On the ambient part $\langle 1, H, H^2, H^3, H^4 \rangle$, in the basis $(1, H, H^2, w_3, w_4)$ with

$$w_3 = H^3 - 21q, \qquad w_4 = H^4 - 6qH,$$

one has $\boldsymbol{\kappa} = 3H\star$ with characteristic polynomial

$$\lambda^2\big(\lambda^3 - 729\,q\big).$$

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On the ambient part $\langle 1, H, H^2, H^3, H^4\rangle$, in the basis $(1, H, H^2, w_3, w_4)$ with

$$w_3 = H^3 - 21q, \qquad w_4 = H^4 - 6qH,$$

one has $\boldsymbol{\kappa} = 3H\star$ with characteristic polynomial

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- $\lambda = 0$: on the ambient part, a single 2×2 nilpotent block, with

$$\ker \boldsymbol{\kappa} = \langle w_4\rangle, \qquad \ker \boldsymbol{\kappa}^2 = \langle w_3, w_4\rangle.$$

The atom of a cubic fourfold, II: the whole thing

By degree reasons $\boldsymbol{\kappa}$ annihilates $H^4_{\mathrm{pr}}(X)$. So on all of $H^*(X)$ the eigenvalue 0 has

$$\text{algebraic multiplicity } 24, \qquad \text{geometric multiplicity } 23,$$

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with Hodge polynomial, coefficient by coefficient (w_3, w_4 are algebraic, of types $(3,3)$ and $(4,4)$):

$$P_{\mathcal{A}_0}(t) \;=\; \underbrace{1}_{h^{3,1}}\, t^2 \;+\; \underbrace{(20+2)}_{h^{2,2}_{\mathrm{pr}}+\langle w_3,w_4\rangle} \;+\; \underbrace{1}_{h^{1,3}}\, t^{-2} \;=\; t^2+22+t^{-2},$$

so $\operatorname{lev}(\mathcal{A}_0)=2$ and $\nu(\mathcal{A}_0)=1$.

The equivariant theorem

Theorem 15

Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold with a generically free action of a finite group G by projectively linear transformations. If the G-action is linearizable, $X \sim_G \mathbb{P}(W)$, then there exist a K3 surface Σ carrying a G-action — a single K3, not a G-orbit of several — and a G-equivariant isomorphism of $\mathbb{Q}$-Hodge structures

$$H^4(X, \mathbb{Q})_{\mathrm{pr}} \simeq H^2(\Sigma, \mathbb{Q})(-1).$$

Proof, I: the atom is equivariant, and the bookkeeping

Step 1 (the atom). Since $\mathrm{Pic}(X) = \mathbb{Z}H$, the class H is G-invariant, hence so are $w_3 = H^3 - 21q$ and $w_4 = H^4 - 6qH$. Place the basepoint in $B_X^{G\times\mathrm{Hod}}$: there $\boldsymbol{\kappa}_b$ commutes with G, so every generalized eigenspace is G-stable and

$$\mathcal{A}_0 = \langle w_3, w_4 \rangle \oplus H^4(X, \mathbb{Q})_{\mathrm{pr}}$$

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$$X = X_0 \leftrightarrow X_1 \leftrightarrow \cdots \leftrightarrow X_m = \mathbb{P}(W),$$

with smooth G-invariant centers, each of dimension ≤ 2 and each a disjoint union of G-orbits of smooth connected subvarieties. Centers occur on both sides, so additivity is an *equation*, not a one-way sum: writing $\mathcal{S}_\pm$ for the centers blown up on the X- resp. $\mathbb{P}(W)$-side, and summing ν over all atoms at $\lambda = 0$,

$$\nu(\mathcal{A}_0) \; + \sum_{Z\in\mathcal{S}_-} \nu(Z) \; = \; \underbrace{\textstyle\sum_{\text{atoms of } \mathbb{P}(W)} \nu}_{=0} \; + \sum_{Z\in\mathcal{S}_+} \nu(Z).$$

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Hence $\sum_{\mathcal{S}_+} \nu(Z) \;=\; 1 + \sum_{\mathcal{S}_-} \nu(Z) \;\geq\; 1$.

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So among all components of all centers, on the X-side, there is exactly one — call it Z° — with $h^{2,0} \neq 0$; it is a surface, and $\nu(Z^\circ) = 1$. On the $\mathbb{P}(W)$-side there are none.

Proof, III: why Z° is G-invariant

Step 4. Recall that a G-invariant center need not be connected: it is a disjoint union of G-orbits of smooth connected subvarieties. So a priori the K3-type contribution could come from a center

$$\Sigma_1 \sqcup \cdots \sqcup \Sigma_k, \qquad G \text{ permuting the } \Sigma_i \text{ transitively},$$

and one would obtain only the *induced* Hodge structure $\mathrm{Ind}_H^G H^2(\Sigma_0, \mathbb{Q})(-1)$ with $H = \mathrm{Stab}_G(\Sigma_0)$[3]. Ruling this out is the whole content of the step.

[3]Here $\mathrm{Ind}_H^G V = \mathbb{Q}[G] \otimes_{\mathbb{Q}[H]} V \cong \bigoplus_{gH \in G/H} g \otimes V$ carries the Hodge structure of V on each of the $[G : H]$ summands, with G permuting them; since G permutes summands of a fixed bidegree, each g is a morphism of Hodge structures and $(\mathrm{Ind}_H^G V)^{p,q} = \mathrm{Ind}_H^G (V^{p,q})$.

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2. each $g \in G$ acts by an isomorphism of varieties, hence preserves $h^{2,0}$, and therefore preserves the property "$h^{2,0} \neq 0$" and the value of ν;
3. consequently G acts by permutations on

$$\mathcal{K} := \{\text{components with } h^{2,0} \neq 0\}.$$

By Step 3, $\mathcal{K} = \{Z^\circ\}$ has one element $\Longrightarrow$ $g(Z^\circ) = Z^\circ \forall g$.

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Proof, IV: Z° is a K3, equivariantly

Step 5(a): the minimal model of Z° is unique. Since $h^{2,0}(Z^\circ) \neq 0$ and $h^{2,0}$ is a birational invariant, Z° is not ruled and not rational (both have $h^{2,0} = 0$).

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$$c_1(K_\Sigma) = 0, \qquad h^{2,0}(\Sigma) = 1, \qquad h^1(\Sigma) = 0.$$

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Thank you