G-equivariant atoms

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Setup

In this talk:

- $X/\mathbb{C}$ always stands for a smooth projective variety over $\mathbb{C}$ of dimension n.
- $G < \mathrm{Aut}(X)$ always means a finite subgroup of automorphisms.
- X is said to be a G-**variety** if G acts generically freely on it.
- A pair (X, G) is said to be **non-G-linearizable** if there is no linear G-action on $\mathbb{P}^n$ and a G-equivariant birational map between X and $\mathbb{P}^n$.

The main theorem for the day

Theorem (C–Katzarkov–Kontsevich)

For any multi-degree hypersurface $X := X(1,1,1,1) \subset (\mathbb{P}^1)^4$ *invariant under the* $\mathbb{Z}/2$*-action in* $(\mathbb{P}^1)^4$ *swapping the first and second, and the third and fourth* $\mathbb{P}^1$*-factors,* $(X, \mathbb{Z}/2)$ *is non-*$\mathbb{Z}/2$*-linearizable.*

The same holds for $(X \times \mathbb{P}^1, \mathbb{Z}/2)$ *where we consider the trivial* $\mathbb{Z}/2$*-action on the* $\mathbb{P}^1$*-factor.*

- This shall be proven by the introduction of G-**equivariant atoms** (actually a coarser version of them).
- To arrive at the proper definition, we first require a brief digression.

Basic setup (Input data)

Set $V := H^*(X, \mathbb{Q})$ as the rational Betti cohomology, considered as a **super vector space**.

- $\mathbb{Z}/2$-**grading:** $V = V_{\bar{0}} \oplus V_{\bar{1}}$ by parity of cohomological degree.
- **Poincaré pairing:** For a basis $\{\Delta_a\}$, we define the non-degenerate graded-symmetric pairing:

$$\eta_{ab} := \int_X \Delta_a \cup \Delta_b$$

- **Coordinates:** Let $\{t_a\}$ be formal dual super coordinates with parity $|t_a| = |\Delta_a|$.

Novikov Rings

Let $\overline{\mathrm{NE}}(X,\mathbb{Z})$ be the monoid of effective curve classes $\beta \in H_2(X,\mathbb{Z})/\mathrm{tor}$.

- **Small Novikov ring:**

$$\mathcal{N}_X := \left\{ \sum_\beta c_\beta q^\beta \mid c_\beta \in \mathbb{Q} \right\}$$

- **Big Novikov ring:**

$$\mathrm{Nov}_X := \mathcal{N}_X \widehat{\otimes}_{\mathbb{Q}} \widehat{\mathrm{Sym}}(V^*) \cong \mathcal{N}_X[[t_a^{\mathrm{even}}]] \otimes_{\mathbb{Q}} \Lambda(t_a^{\mathrm{odd}}).$$

Gromov-Witten Potential

The **genus-zero Gromov-Witten potential** $\Phi_X \in \mathrm{Nov}_X$ is defined as:

$$\Phi_X(\boldsymbol{t}, q) = \Phi_{\mathrm{cl}}(\boldsymbol{t}) + \Phi_{\mathrm{q}}(\boldsymbol{t}, q)$$

- **Classical part:** Intersection theory on X

$$\Phi_{\mathrm{cl}}(\boldsymbol{t}) := \frac{1}{3!} \sum_{a,b,c} \left(\int_X \Delta_a \cup \Delta_b \cup \Delta_c \right) t_a t_b t_c$$

- **Quantum correction:** Sum over non-zero curve classes β and n-point (genus zero) Gromov-Witten invariants $\langle \Delta_{a_1}, \ldots, \Delta_{a_n} \rangle_{n,\beta}$.

The Big Quantum Product

The potential Φ_X defines the **big quantum product** $\star$ on $V_{\text{big}} := V \otimes_{\mathbb{Q}} \text{Nov}_X$:

$$\Delta_a \star \Delta_b := \sum_{c,d} \frac{\partial^3 \Phi_X}{\partial t_a \partial t_b \partial t_d} \eta^{dc} \Delta_c$$

- The product is graded-commutative and associative (equivalent to the **WDVV equations**).
- This structure defines the Big Quantum Cohomology algebra $QH^*(X) := (V_{\text{big}}, \star, \eta)$.

Small Quantum Cohomology

Definition

Evaluating the big quantum product at $\boldsymbol{t} = 0$ yields the **small quantum product** $\star_{\text{small}}$ on $V_{\text{small}} := V \otimes_{\mathbb{Q}} \mathcal{N}_X$:

$$\Delta_a \star_{\text{small}} \Delta_b := \sum_{c,d} \left(\sum_{\beta} \langle \Delta_a, \Delta_b, \Delta_d \rangle_{3,\beta} q^{\beta} \right) \eta^{dc} \Delta_c$$

The resulting algebra $QH^*_{\text{small}}(X)$ is graded-associative and graded-commutative.

Example: Quantum Cohomology of $\mathbb{P}^2$

Let $X = \mathbb{P}^2$ with basis $\{\Delta_0, \Delta_1, \Delta_2\} = \{\mathbf{1}, H, P\}$ and dual coordinates $\boldsymbol{t} = (t_0, t_1, t_2)$.

Gromov-Witten Potential

$$\Phi_X(\boldsymbol{t}, q) = \frac{1}{2} t_0^2 t_2 + \frac{1}{2} t_0 t_1^2 + \sum_{d=1}^{\infty} N_d \frac{t_2^{3d-1}}{(3d-1)!} q^d e^{dt_1}$$

where N_d is the number of degree d rational curves through $3d-1$ points.

Big Quantum Product Relations

Defining $\Gamma_{ijk} := \partial_{t_i} \partial_{t_j} \partial_{t_k} \Phi_{\mathrm{q}} \in \mathrm{Nov}_X$:

$$H \star H = P + \Gamma_{111} H + \Gamma_{112} \mathbf{1}$$

$$H \star P = \Gamma_{112} H + \Gamma_{122} \mathbf{1}$$

$$P \star P = \Gamma_{122} H + \Gamma_{222} \mathbf{1}$$

Convergence of the Gromov-Witten Potential

While proven for certain classes, it is expected in general that the GW potential formal power series, and hence the structure constants, have a **positive radius of convergence**.

Examples

- For $X = \mathbb{P}^1$, the potential is an entire function.
- For $X = \mathbb{P}^2$, the invariants grow as bounded by:

$$\lim_{d\to\infty} \left(\frac{N_d}{(3d)!} \right)^{1/d} = A \approx 0.1380093\ldots$$

Geometric Interpretation of the Quantum Product

- The big quantum product deforms the commutative associative unital algebra structure on the super vector space $H^*(X, \mathbb{Q})$ over parameters $(\boldsymbol{t}, q) \in \mathrm{Spf}(\mathrm{Nov}_X)$.
- This structure can be interpreted as a multiplication on the *tangent bundle* to a specific supermanifold whose super-dimension matches the rank of the cohomology.
- **Example:** Let $X = \mathbb{P}^2$. Set $q \in \mathbb{C}^\times$ with $|q| < (0.138009\ldots)^{-1}$. This specialization yields a germ of an analytic function in an open domain of $H^*(X, \mathbb{C})$ with coordinates t_0, t_1, t_2, endowed with the flat holomorphic metric:

$$dt_1 \otimes dt_1 + dt_0 \otimes dt_2 + dt_2 \otimes dt_0.$$

The **WDVV equations** imply that the third covariant derivative of this function (raising one index) gives an associative commutative unital product on the tangent bundle.

Redundancy and the Divisor Axiom

There is a redundancy in the parameters $(\boldsymbol{t}, q)$.
Assume the basis $\{\Delta_a\}_{a\in I}$ contains an integer basis $\Delta_1, \ldots, \Delta_r$ of the Néron-Severi lattice:

$$\mathrm{NS}(X) := \big(H^2(X,\mathbb{Z})/\text{torsion}\big) \cap H^{1,1}(X) \subset H^2(X,\mathbb{Q})$$

where $r = \operatorname{rk} \mathrm{NS}(X)$.
The additive generators q^β of $\mathcal{N}_X$ can be written as Laurent monomials in variables $q_1, \ldots, q_r$:

$$q^\beta = \prod_{i=1}^{r} q_i^{\langle \beta, \Delta_i \rangle}$$

Claim: The dependence of the quantum product on the dual variables $t_1, \ldots, t_r$ can be completely subsumed into the variation of parameters q.

The Reduced Big Novikov Ring

Define the *reduced big Novikov ring* $\mathrm{Nov}_X^{\mathrm{red}} \subset \mathrm{Nov}_X$ as the image of the injective homomorphism:

$$\mathcal{N}_X[[(t_a^{\mathrm{even}})_{a\notin\{1,\dots,r\}}]] \otimes_{\mathbb{Q}} \Lambda(t_a^{\mathrm{odd}}) \to \mathrm{Nov}_X$$

given on generators by:

$$q^\beta \mapsto q^\beta \prod_{i=1}^{r} e^{t_i\langle\beta,\Delta_i\rangle}$$
$$t_a \mapsto t_a \quad \forall a \notin \{1,\dots,r\}$$

The *divisor axiom* implies that the structure constants for the big quantum product actually belong to the subring $\mathrm{Nov}_X^{\mathrm{red}}$.

The Logarithmic Tangent Bundle

Using $\mathrm{Nov}_X^{\mathrm{red}}$, the big quantum product becomes a product on the *logarithmic tangent bundle* to $\operatorname{Spf} \mathrm{Nov}_X^{\mathrm{red}}$.
We identify the generators as derivations on $\mathcal{N}_X$:

- For $a \notin \{1, \ldots, r\}$:

$$\Delta_a \mapsto \frac{\partial}{\partial t_a}$$

- For $a = i \in \{1, \ldots, r\}$ (the NS generators):

$$\Delta_i \mapsto q_i \frac{\partial}{\partial q_i}$$

where $q_i \frac{\partial}{\partial q_i}(q^\beta) := \langle \beta, \Delta_i \rangle q^\beta$.

Ensuring Convergence: Rigid Analytic Geometry

Since the convergence of the GW potential is hypothetical in general, we ensure convergence by introducing an algebraically closed non-Archimedean field $\Bbbk$ of characteristic zero (e.g., Puiseux series):

$$\Bbbk := \bigcup_{n \geq 1} \overline{\mathbb{Q}}(\!(T^{1/n})\!)$$

with $|T|_{\Bbbk} = c < 1$.
We consider homomorphisms $\phi_{\boldsymbol{q}} : \overline{\mathrm{NE}}(X, \mathbb{Z}) \to \Bbbk^{\times}$ and maps $\phi_{\boldsymbol{t}}$ for non-NS even coordinates. By enforcing Kähler bounds $|\phi_{\boldsymbol{q}}(\beta)|_{\Bbbk} = \exp(-\langle \beta, [\omega^{1,1}] \rangle)$ and $|\phi_{\boldsymbol{t}}(a)|_{\Bbbk} < 1$, the structure constants converge adically.

The A-model Manifold

Definition

For a complex smooth projective variety X, the *A-model manifold* is an analytic supermanifold B_X over $\Bbbk$ defined as the product $U_1 \times U_2 \times U_{\mathrm{odd}}$:

- U_1 **(Tube Domain):** $\mathrm{pr}^{-1}(-\mathcal{U}) \subset (\mathbb{G}^{\mathrm{an}}_{m,\Bbbk})^r$, where $\mathcal{U}$ is the open cone of ample classes in $\mathrm{NS}(X) \otimes \mathbb{R}$.
- U_2 **(Open Discs):** Polydisc for the remaining even coordinates:

$$\{(t_a^{\mathrm{even}})_{a \notin \{1,\ldots,r\}} \mid |t_a^{\mathrm{even}}|_{\Bbbk} < 1\}$$

- U_{odd}**:** Purely odd coordinates with function algebra $\Bbbk \otimes_{\mathbb{Q}} \Lambda(t_a^{\mathrm{odd}})$.

Example: A-model Manifold for $\mathbb{P}^2$

Let $X = \mathbb{P}^2$ with basis $\{\mathbf{1}, H, P\}$. The Néron-Severi group $\mathrm{NS}(\mathbb{P}^2) \otimes \mathbb{R}$ has rank $r = 1$, generated by $\Delta_1 = H$.

Components of $B_{\mathbb{P}^2}$

- **Tube Domain (U_1):** The ample cone is $\mathbb{R}_{>0}$, so U_1 is the punctured open unit disc in $\mathbb{A}^{1,\mathrm{an}}_{\Bbbk}$, given by $0 < |q|_{\Bbbk} < 1$.
- **Open Polydisc (U_2):** The remaining even coordinates ($\mathbf{1}$ and P) form an open unit bidisc in $\Bbbk^2$:

$$U_2 = \{(t_0, t_2) \in \Bbbk^2 \mid |t_0|_{\Bbbk} < 1, |t_2|_{\Bbbk} < 1\}.$$

- **Odd Coordinates (U_{odd}):** Since cohomology is entirely even, U_{odd} is just a point.

Affine Structures and Frobenius Manifolds

The coordinate identification endows the A-model manifold B_X with an **affine structure** (a torsion-free flat connection on its tangent bundle) and a compatible flat metric.

Historical Context

- **Frobenius manifolds:** Traditionally defined as having an affine structure, a compatible flat metric, and a potential whose third covariant derivative yields the product $\star$.
- **Our perspective:** It is more convenient to define the quantum product strictly on the tangent bundle, such that in the formal neighborhood of any point, it is the third derivative of a potential (well-defined up to a polynomial of degree ≤ 2).

F-manifolds and the Euler Field

C. Hertling and Yu. Manin weakened the Frobenius structure by dropping the metric and affine structure, keeping only the product $\star \in \Gamma(M, \operatorname{Sym}^2 T^*_M \otimes T_M)$, resulting in F-**manifolds**.

F-bundles and the Euler Field

Katzarkov–Kontsevich–Pantev–Yu introduces F-**bundles**, which induce an F-manifold structure alongside an Euler vector field Eu satisfying:

$$\operatorname{Lie}_{\mathsf{Eu}}(\star) = \star$$

For Gromov-Witten invariants, Eu at $b \in B_X$ is canonically given by:

$$\mathsf{Eu}_b := c_1(T_X) + \sum_{\deg \Delta_a \neq 2} \left(\frac{2 - \deg \Delta_a}{2} \right) t_a(b) \Delta_a \ \in \ T_b B_X$$

Pre-F-manifolds with Euler Field

On the purely even subvariety $B_{X,\mathrm{small}} \subset B_X$ (setting non-NS coordinates $t_a = 0$), the Euler field simplifies to $\mathrm{Eu}_b = c_1(T_X)$.

Definition (Pre-F-manifold)

A *pre-F-manifold with Euler field* is an analytic supermanifold over $\Bbbk$ with an even bilinear operation $\star$ on the tangent bundle, and an even vector field Eu such that:

- $\star$ gives an associative, commutative, unital product.
- $\mathrm{Lie}_{\mathrm{Eu}}(\star) = \star$.

Example ($\mathbb{P}^2$): A-model manifolds are natural pre-F-manifolds. A straightforward computation yields the Euler field:

$$\mathrm{Eu} = 3q\partial_q + t_0\partial_0 - t_2\partial_2$$

Disjoint Unions and Products

Disjoint Unions and Thom-Sebastiani

If a variety is a disjoint union $X \sqcup Y$, its Gromov-Witten potential splits as a **Thom-Sebastiani sum** of the potentials of the factors:

$$\Phi_{X \sqcup Y} = \Phi_X \oplus \Phi_Y$$

Geometric consequence: At the level of A-model manifolds, this implies that the manifold of a disjoint union is canonically the Cartesian product of the manifolds of its components:

$$B_{X \sqcup Y} \cong B_X \times B_Y$$

Quantum Leray-Hirsch

Theorem (Quantum Leray-Hirsch, Iritani-Koto 2024)

Let $\pi\colon P = \mathbb{P}(V) \to X$ be a projective bundle of rank r over a smooth projective variety X.

There exists a canonical analytic identification between open nonempty connected domains in B_P and in the product of r copies of B_X:

$$B_P \supset U \xleftrightarrow{\sim} V \subset \underbrace{B_X \times \cdots \times B_X}_{r \text{ times}} \cong B_{X \sqcup \cdots \sqcup X}$$

This identification rigidly preserves both the **quantum multiplication** $\star$ *and the* **Euler field** Eu.

Classical Cohomology of a Blowup

We now turn to an analogous statement for blowups.

Setup: Fix a smooth projective variety X and a smooth subvariety $Z \subset X$ of codimension $r \geq 2$. Let $\pi\colon \widetilde{X} \to X$ be the blowup of X along Z.

- Exceptional divisor: $j\colon E \hookrightarrow \widetilde{X}$, where $E \cong \mathbb{P}(N_{Z/X})$.
- Divisor class: $\zeta = [E] \in H^2(\widetilde{X}, \mathbb{Q})$ and $h := -j^*\zeta \in H^2(E, \mathbb{Q})$.

Classical Decomposition

It is a well-known fact that the classical cohomology splits as:

$$H^*(\widetilde{X}, \mathbb{Q}) \cong \pi^* H^*(X, \mathbb{Q}) \oplus \bigoplus_{i=1}^{r-1} j_* \left(h^{i-1} \cup \pi|_E^* H^*(Z, \mathbb{Q}) \right) \tag{1}$$

Quantum Cohomology of a Blowup

Theorem (Iritani 2023)

Let $\widetilde{X}$ be the blowup of a smooth projective variety X along a smooth subvariety $Z \subset X$ of codimension $r \geq 2$.

There exists a canonical analytic identification between open nonempty connected domains in $B_{\widetilde{X}}$ and in the product of B_X and $r-1$ copies of B_Z:

$$B_{\widetilde{X}} \supset U \xleftrightarrow{\sim} V \subset B_X \times \underbrace{B_Z \times \cdots \times B_Z}_{r-1 \text{ times}} \cong B_{X \sqcup Z \sqcup \cdots \sqcup Z}$$

As before, this identification strictly preserves the quantum multiplication $\star$ and the Euler field Eu.

(Coarse) Hodge Atoms

- **Tate Twist:** $\mathbb{Q}(n)$ has underlying $\mathbb{Q}$-vector space $(2\pi\mathbf{i})^n\mathbb{Q} \subset \mathbb{C}$ equipped with a pure Hodge structure of type $(-n,-n)$. For a Hodge structure V of weight k, its n-th Tate twist is $V(n) := V \otimes_{\mathbb{Q}} \mathbb{Q}(n)$, of weight $k-2n$.
- **Polarizable Hodge Structure:** A weight k Hodge structure V is polarizable if there is a morphism $Q : V \otimes_{\mathbb{Q}} V \to \mathbb{Q}(-k)$ satisfying the Hodge-Riemann bilinear relations.
- **Tannakian Category:** Polarizable pure $\mathbb{Q}$-Hodge structures form a neutral Tannakian category $\mathcal{HS}$. By Tannakian formalism, $\mathcal{HS}$ is equivalent to representations of the *Mumford-Tate group*, MT.

The Hodge Group and its Action

The Mumford-Tate group MT has two canonical homomorphisms:

$$\mathbb{G}_{m,\mathbb{Q}} \xrightarrow{f_1} \mathrm{MT} \xrightarrow{f_2} \mathbb{G}_{m,\mathbb{Q}}$$

- The **Hodge group** is $\mathrm{Hod} := \ker(f_2 : \mathrm{MT} \to \mathbb{G}_{m,\mathbb{Q}})$.
- A distinguished central element $\epsilon := f_1(-1) \in \mathrm{Hod}$ of order 2 acts on weight k structures as $(-1)^k$.
- Hod naturally acts on $\mathbb{Z}/2$-graded polarizable pure Hodge structures.

For a complex smooth projective variety X, its Betti cohomology $H^*(X, \mathbb{Q})$ carries a natural $\mathbb{Z}/2$-graded polarizable Hodge structure, admitting a Hod action.

The Non-Archimedean Hodge Group and the A-model Manifold

- Consider the non-Archimedean Hodge group: $\mathrm{Hod}_{\Bbbk} := \mathrm{Hod} \otimes_{\mathbb{Q}} \Bbbk$.
- **Basis Choice & Submodules:** Recall we choose a basis where elements $\Delta_1, \ldots, \Delta_r$ span the Néron-Severi group. We assume the remaining even coordinates form a basis of a Hod-submodule in $H^*(X)$.
- This basis choice naturally splits the action:
  - **On U_1:** The classes $\Delta_1, \ldots, \Delta_r$ are algebraic, meaning they are fixed points under the Hod-action. Consequently, the action on the tube domain U_1 is strictly **trivial**.
  - **On U_2 and U_{odd}:** The action follows the coordinates of the submodule. For example, the central element ϵ acts on the remaining coordinates t_a (dual to Δ_a of weight k) as $\epsilon \cdot t_a = (-1)^k t_a$.

Hodge Fixed Locus and Invariance

- Gromov-Witten correlators and the Poincaré pairing are morphisms in $\mathcal{HS}$, thus equivariant under MT and Hod.
- Any $g \in \mathrm{Hod}(\Bbbk)$ acts as an automorphism of the pre-F-manifold B_X, preserving the quantum product $\star$ and the Euler field Eu.
- The **Hodge fixed locus** B_X^{Hod} consists of points invariant under $\mathrm{Hod}(\Bbbk)$, geometrically where non-(p,p) coordinates vanish.

Decompositions of the Tangent Space

At any point $b \in B_X^{\mathrm{Hod}}$, the tangent space $T_b B_X$ inherits two canonical linear decompositions:

1. **Tannakian Decomposition:**

$$T_b B_X = \bigoplus_{\gamma \in \mathrm{Irrep}(\mathsf{Hod}_{\Bbbk})} (T_b B_X)_\gamma$$

   where $(T_b B_X)_\gamma$ are isotypic components for the $\mathsf{Hod}_{\Bbbk}$-action.

2. **Spectral Decomposition:** Using the Euler operator $\boldsymbol{\kappa}_b(v) := \mathrm{Eu}_b \star_b v$,

$$T_b B_X = \bigoplus_{\lambda \in \mathrm{Spec}(\boldsymbol{\kappa}_b)} \mathcal{V}_{\lambda, b}$$

   where $\mathcal{V}_{\lambda,b}$ are generalized eigenspaces.

The Generic Locus and Coarse Hodge Atoms

- Since $g \cdot \mathrm{Eu}_b = \mathrm{Eu}_b$ for $b \in B_X^{\mathrm{Hod}}$, the Euler operator $\boldsymbol{\kappa}_b(v) := \mathrm{Eu}_b \star_b v$ is a morphism of $\mathrm{Hod}_{\Bbbk}$-representations.
- Consequently, each generalized eigenspace $\mathcal{V}_{\lambda,b}$ is a well-defined $\mathrm{Hod}_{\Bbbk}$-submodule.
- **The Generic Locus:** Distinct eigenspaces merge when eigenvalues collide. Let $N(b)$ be the number of distinct roots of the characteristic polynomial $P_b(\lambda)$ of $\boldsymbol{\kappa}_b$. The generic locus is defined as:

$$\mathcal{U}_X := \{b \in B_X^{\mathrm{Hod}} \mid N(b) = N_{\max}\}$$

Definition: Coarse Hodge Atom

A **coarse Hodge atom** at a point b is the isomorphism class $[\mathcal{V}_{\lambda,b}]$ of the $\mathrm{Hod}_{\Bbbk}$-representation associated to an eigenvalue $\lambda \in \mathrm{Spec}(\boldsymbol{\kappa}_b)$.

Algebraic Structure of Coarse Atoms

The tangent space $T_b B_X$ at any point is inherently a commutative, associative algebra with unit $\mathbf{1}$ under the quantum product $\star_b$.

The Commuting Property

The Euler operator is simply left-multiplication by the Euler vector field:

$$\boldsymbol{\kappa}_b(v) = \mathrm{Eu}_b \star_b v$$

Because $\star_b$ is associative and commutative, multiplication by Eu_b commutes with multiplication by any other element $w \in T_b B_X$:

$$\boldsymbol{\kappa}_b(w \star_b v) = \mathrm{Eu}_b \star_b (w \star_b v) = w \star_b (\mathrm{Eu}_b \star_b v) = w \star_b \boldsymbol{\kappa}_b(v)$$

Consequence: The generalized eigenspaces $\mathcal{V}_{\lambda,b}$ are **ideals** of the algebra $(T_b B_X, \star_b)$. Consequently, the tangent space decomposes as a direct sum of rings, meaning every coarse Hodge atom $\mathcal{V}_{\lambda,b}$ intrinsically inherits the structure of a **unital commutative associative algebra** over $\Bbbk$.

Rigidity of Coarse Atoms

Lemma (KKPY)

For any rigid point $b \in \mathcal{U}_X$, the $\mathsf{Hod}_{\Bbbk}$-representation on the eigenspace $\mathcal{V}_{\lambda,b}$ is induced by extension of scalars from a finite-dimensional $\overline{\mathbb{Q}}$-linear representation E^{α} of $\mathsf{Hod}_{\overline{\mathbb{Q}}}$. This isomorphism class is **independent of** *b.*

Consequence: Each coarse atom α corresponds to a well-defined module $E^{\alpha} \in \mathrm{Rep}_{\overline{\mathbb{Q}}}(\mathsf{Hod}_{\overline{\mathbb{Q}}})$.

Numerical Invariants

After complexification $E^\alpha_{\mathbb{C}}$, the torus $\mathbb{G}_{m,\mathbb{C}} \subset \mathrm{Hod}_{\mathbb{C}}$ induces the *Hochschild grading* $(p-q)$.

Definition

1. **Dimension of Hodge classes (ρ_α):** The complex dimension of the invariant subspace (note $\rho_\alpha \geq 1$ since the algebraic unit $\mathbf{1}$ is invariant):
$$\rho_\alpha := \dim_{\mathbb{C}} \left(E^\alpha_{\mathbb{C}}\right)^{\mathrm{Hod}_{\mathbb{C}}}$$
2. **Hodge polynomial ($P_\alpha(t)$):** The generating function for the Hochschild weight spaces:
$$P_\alpha(t) := \sum_{k\in\mathbb{Z}} \dim_{\mathbb{C}} \left(\left(E^\alpha_{\mathbb{C}}\right)_k\right) t^k$$

Atomic Collisions and Blowups

Outside the generic locus $\mathcal{U}_X$, eigenvalues can collide. When eigenvalues $\lambda_1, \ldots, \lambda_k$ merge into λ_{spec} at b_{spec}, their representations merge additively:

$$[\mathcal{V}_{\lambda_{\text{spec}}, b_{\text{spec}}}] = \bigoplus_{i=1}^{k} [\mathcal{V}_{\lambda_i, b_{\text{gen}}}]$$

Decomposition under Blowups

Recall the local identification for the blowup $\widetilde{X}$ of X along Z (codim r):

$$B_{\widetilde{X}} \supset U \xleftrightarrow{\sim} V \subset B_X \times \underbrace{B_Z \times \cdots \times B_Z}_{r-1 \text{ times}}$$

This preserves the Euler field and $\mathrm{Hod}_{\mathbb{k}}$-action. Thus, the coarse atoms of $\widetilde{X}$ are the disjoint union of the atoms of X and $r-1$ copies of the atoms of Z.

Obstructions to Rationality

Theorem (Katzarkov–Kontsevich–Pantev–Yu)

If a smooth complex projective variety X of dimension $d \geq 2$ contains a Hodge atom α in its decomposition that cannot originate from a variety of dimension $\leq d-2$, then X is **non-rational**.

G-equivariant Atoms and Invariants

For a G-variety X, the product $G \times \mathrm{Hod}$ acts on the A-model manifold B_X.

Coarse G-Atoms

Restricting to the fixed locus $B_X^{G\times \mathrm{Hod}}$, a coarse G-atom α is the isomorphism class of the $G \times \mathrm{Hod}_{\Bbbk}$-representation on the generalized eigenspace $\mathcal{V}_{\lambda,b}$.

Alongside the dimension of Hodge classes ρ_α and the Hodge polynomial $P_\alpha(t)$, we introduce a crucial equivariant invariant:

Definition

The *dimension of G-invariant Hodge classes*, denoted ρ_α^G, tracks invariants under the joint action:

$$\rho_\alpha^G := \dim_{\mathbb{C}} \left(E_{\mathbb{C}}^\alpha\right)^{G\times \mathrm{Hod}_{\mathbb{C}}}$$

Obstructions to G-Linearizability

Theorem (C–Katzarkov–Kontsevich)

Let X be a G-variety of dimension $d \geq 2$. If X contains a coarse G-atom $\alpha \notin \mathrm{HAtoms}^G_{dim \leq d-2}$, then X **is not G-birationally equivalent to $\mathbb{P}^d$ with a linearizable G-action.**

Classification of Low-Dimensional G-Atoms

To use this obstruction, we classify atoms from dimensions ≤ 1. They arise *only* from:

1. An orbit G/H of a point.
2. A product $G \times_H C$, where C is a curve of genus $g \geq 1$.

Key Constraint: $\rho^G_\alpha \in \{1, 2\}$. Furthermore, if $P_\alpha(t)$ is constant, then $\rho^G_\alpha = 1$.

Setup and the Main Theorem

Let $X := X(1,1,1,1) \subset (\mathbb{P}^1)^4$. It is **rational** (the blowup of $(\mathbb{P}^1)^3$ along an elliptic curve).

Cohomology and the G-action

The algebraic classes have dimensions $(1,4,4,1)$. The $G = \mathbb{Z}/2$ action permuting factors $(1 \leftrightarrow 2)$ and $(3 \leftrightarrow 4)$ restricts $\rho^G_{total} = 6$ invariant Hodge classes.

Theorem (C–Katzarkov–Kontsevich)

There is **no** $\mathbb{Z}/2$*-linearizable action on* $\mathbb{P}^3$ *making it* $\mathbb{Z}/2$*-equivariantly birational to* X*.*

Proof

Small quantum multiplication implies X has an atom with invariants:

$$(\rho, \rho^G) = (5, 3) \quad \text{and non-algebraic piece} \quad h^{2,1} = 1$$

1. The $h^{2,1} = 1$ term requires the elliptic curve atom, which contributes $(2, 2)$.
2. $$(\rho, \rho^G)_{\text{remaining}} = (5, 3) - (2, 2) = (\mathbf{3}, \mathbf{1}).$$

   This remainder **must** be formed by the available basic point atoms, $(1, 1)$ and $(2, 1)$:
   $$a(1, 1) + b(2, 1) = (3, 1).$$

A dichotomy

Corollary (C–Katzarkov–Kontsevich)

For any subgroup $G < S_4$ acting on $X(1,1,1,1)$ induced from a transitive action on $\{1,2,3,4\}$, exactly one of the following holds:

(a) *The G-action on the set $\{1,2,3,4\}$* **has a fixed point**. *In this case, X is G-birationally equivalent to $\mathbb{P}^3$ with a G-action.*

(b) *The G-action on the set $\{1,2,3,4\}$* **has no fixed points**. *In this case, X is* **not** *G-birationally equivalent to $\mathbb{P}^3$ with a G-action.*

Proof idea for (b): If there are no fixed points, the group G must contain (up to conjugation) the specific $\mathbb{Z}/2$ subgroup swapping $(1 \leftrightarrow 2)$ and $(3 \leftrightarrow 4)$.