Torus actions on the moduli space of stable maps and GKM varieties

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The Theory of Atoms: Motivation

Recent Developments

Katzarkov, Kontsevich, Pantev, and Yu introduced the **"Theory of Atoms"** to study birational geometry problems.

Theorem (Katzarkov-Kontsevich-Pantev-Yu)

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A-model F-bundles

The Setup

We define the non-archimedean field of Puiseux series: $\mathbb{k} := \overline{\mathbb{Q}}((T^{\mathbb{Q}}))$

Input:

- a smooth projective variety $X/\mathbb{C}$
- specifically, its rational Betti cohomology $H := H_B^*(X, \mathbb{Q})$ with the usual $\mathbb{Z}/2$-grading (parity).

Output:

- a base (super)manifold B_X
- a flat holomorphic (super) vector bundle $\mathcal{H} \to B_X \times \mathbb{D}$, where $\mathbb{D}$ is the germ at 0 of a non-archimedean analytic disc (parameter u)
- a meromorphic flat connection ∇ satisfying the regularity conditions:

$$\nabla_{u^2 \frac{\partial}{\partial y}} \text{ and } \nabla_{u\xi}$$

are *regular*, where ξ is a local vector field of B_X.

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Geometric Intuition

1. The Quantum Multiplication

- The third derivatives of the Gromov-Witten potential (contracted with Poincaré pairing) define a multiplication $\star_b$ on $H \otimes_{\mathbb{Q}} \Bbbk$.
- This family, indexed by $b \in B_X$, is the **(big) quantum cohomology**.

2. Canonical Identifications

- The fiber identifies with the tangent space and the algebra:

$$\mathcal{H}_{(b,0)} \cong T_b B_X \cong (H \otimes_{\mathbb{Q}} \Bbbk, \star_b)$$

3. The Euler Vector Field

- A distinguished vector field Eu defines the operator κ acting on the fiber:

$$\kappa := \nabla_{u^2 \frac{\partial}{\partial u}}\Big|_{u=0} = \mathrm{Eu}\star$$

In the picture, B_X^{MT} means the fixed locus by the action of the *Mumford-Tate group* $\mathrm{MT}(\Bbbk)$. Simply, $\mathcal{HS}^{\mathbb{Z}/2} \cong \mathrm{RepMT}$.

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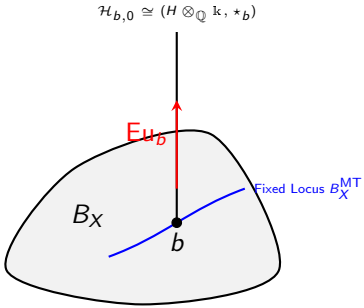
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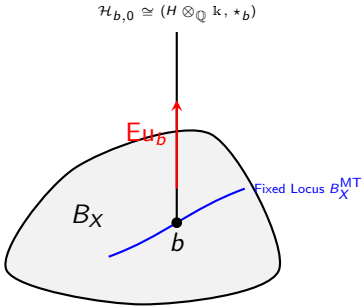
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Defining "Hodge Atoms"

The Spectral Decomposition

- For generic $b \in B_X^{\mathrm{MT}}$, the operator κ is semi-simple.
- Its action induces a splitting of the fiber $\mathcal{H}_{(b,0)}$ into eigenspaces.
- **Definition:** We term these irreducible components *Hodge atoms*.

The Computational Challenge

- To apply the theory, we must explicitly compute these decompositions.
- **Core Task:** Determine the eigenvalues (spectrum) of κ.

The Main Question

Can we obtain **simple combinatorial methods** to compute these Hodge atoms for specific classes of varieties?

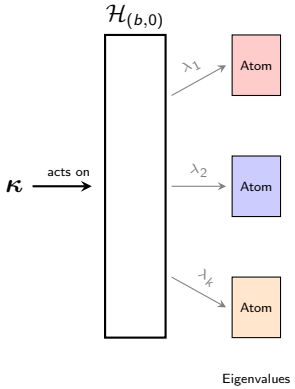
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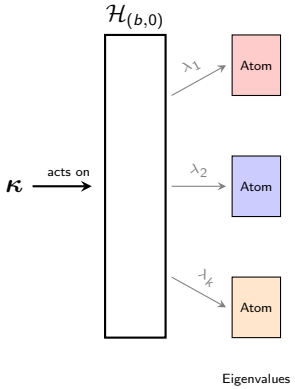
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Defining GKM Varieties

Definition (Algebraic GKM Variety)

Let $X/\mathbb{C}$ be a smooth projective variety equipped with an effective algebraic action of a torus $\mathbf{T} \cong (\mathbb{C}^\times)^r$ (where $r \geq 2$).
We say X is a **GKM space**[a] if:

(a) The fixed point set $X^{\mathbf{T}}$ is finite.

(b) The number of 1-dimensional $\mathbf{T}$-orbits is finite.

(c) X is **equivariantly formal**. That is, if $R := H^*_{\mathbf{T}}(\mathrm{pt}, \mathbb{C})$, we have an isomorphism of R-modules:

$$H^*_{\mathbf{T}}(X, \mathbb{C}) \cong H^*(X, \mathbb{C}) \otimes_{\mathbb{C}} R$$

[a] Goresky-Kottwitz-MacPherson

Reduction of Action

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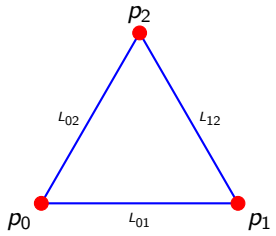
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GKM graphs

1. Projective Space $\mathbb{P}^2$

Construction Rules

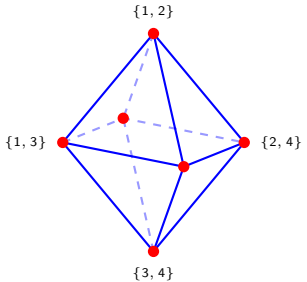
- **Vertices:** Coordinate points p_i (where only $z_i \neq 0$).
- **Edges:** Invariant lines connecting p_i, p_j (where only $z_i, z_j \neq 0$).



2. Grassmannian $Gr(2,4)$

Construction Rules

- **Vertices:** Subsets $I \subset \{1..4\}$ with $|I| = 2$. (Coordinate planes).
- **Edges:** Connect I, J if $|I \cap J| = 1$. (They share one basis vector).



The Moduli Space of Stable Maps

Definition (Stable Map)

The moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$ parametrizes isomorphism classes of **stable maps** $(C, x_1, \ldots, x_n, f)$, where:

- C is a connected, nodal projective curve of arithmetic genus g.
- $x_1, \ldots, x_n$ are distinct non-singular marked points on C.
- $f : C \to X$ is a morphism such that $f_*[C] = \beta \in H_2(X, \mathbb{Z})$.

Stability Condition: The group of automorphisms $\mathrm{Aut}(f)$ respecting the marked points is finite.

Induced Torus Action

If a torus $\mathbf{T}$ acts on X, it naturally induces an action on the moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$ via **post-composition**:

$$t \cdot (C, x_i, f) := (C, x_i, t \circ f), \quad \forall t \in \mathbf{T}.$$

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T-actions on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$

The Induced Action

The torus $\mathbf{T} \cong (\mathbb{C}^*)^{m+1}$ acts on $\mathbb{P}^m$ by scaling coordinates. This induces a canonical action on the moduli space:

$$t \cdot (C, x_i, f) = (C, x_i, t \circ f)$$

The Fixed Point Locus

The fixed locus $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)^{\mathbf{T}}$ is not irreducible.

- Its connected components are indexed by **decorated graphs** Γ.
- **Genus 0 assumption:** Since $g = 0$, these graphs are always **trees**.

Combinatorial Data: The Graph Γ

A graph $\Gamma \in \mathcal{G}_{0,n}(\mathbb{P}^m, d)$ consists of data (V, E, N, j, δ) describing the map limit:

- **Vertices (V):** Connected components mapped to fixed points $p_{j(v)} \in (\mathbb{P}^m)^{\mathbf{T}}$.
- **Edges (E):** Non-contracted rational components connecting $p_{j(v)}$ and $p_{j(v')}$.
- **Markings (N):** The set $\{1, \ldots, n\}$ incident to vertices.
- **Degree (δ):** Map $\delta : E \to \mathbb{Z}_{>0}$ (degree of the cover).

Genus 0 Constraints:

1. Γ is a Tree.
2. All vertices represent rational curves ($\gamma(v) = 0$).

Geometric Correspondence

Let $\pi : (C, x_i) \to \mathbb{P}^m$ be a $\mathbf{T}$-fixed stable map. The geometry is constrained as follows:

1. **Vertices (Contracted):** Components of C contracted to a fixed point p_i.
2. **Edges (Invariant Curves):** Non-contracted components $D \subset C$ must map to invariant lines L_{ij} connecting p_i and p_j.
   - $\pi|_D : D \to L_{ij}$ is a Galois cover ramified only over p_i, p_j.
   - Determined uniquely by the degree $\delta(e)$.

Bijection

$$\left\{\text{Components of } \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)^{\mathbf{T}}\right\} \longleftrightarrow \mathcal{G}_{0,n}(\mathbb{P}^m, d)$$

Structure of the Fixed Locus

For a specific graph Γ, the fixed locus component $\mathcal{Q}_\Gamma$ is a finite quotient.

The Moduli Factor

$$\mathcal{M}_\Gamma = \prod_{v \in V} \overline{\mathcal{M}}_{0,\mathrm{val}(v)}$$

(Note: $\overline{\mathcal{M}}_{0,1}$ and $\overline{\mathcal{M}}_{0,2}$ are points).

The Quotient Stack:

$$\mathcal{Q}_\Gamma \cong \mathcal{M}_\Gamma / A_\Gamma \hookrightarrow \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$$

where A_Γ is the automorphism group of the graph and the edge covers:

$$1 \to \prod_{e \in E} \mathbb{Z}/\delta(e) \to A_\Gamma \to \mathrm{Aut}(\Gamma) \to 1$$

Tautological Classes (Genus 0)

To compute integrals, we need tautological classes on the moduli space.

- **Psi Classes (ψ):** Let $\mathbb{L}_i$ be the cotangent line bundle at the i-th marked point.

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d), \mathbb{Q})$$

- **Hodge Classes (λ):** *Remark:* In the genus $g = 0$ case, the Hodge bundle $\mathbb{E}$ is trivial, so λ classes vanish.

The Virtual Localization Formula

The moduli space is virtually smooth. We compute invariants by summing over graphs Γ.

Theorem (Graber-Pandharipande)

In the localized equivariant Chow ring:

$$[\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)]^{vir} = \sum_{\Gamma \in \mathcal{G}_{0,n}} \frac{1}{|A_\Gamma|} (\tau_\Gamma)_* \left(\frac{[\mathcal{M}_\Gamma]^{vir}}{e(N_\Gamma^{vir})} \right)$$

- Summation is over all decorated graphs Γ.
- $e(N_\Gamma^{vir})$ is the Euler class of the virtual normal bundle (computable explicitly in terms of graph weights).

The Normal Bundle Contribution

Goal: Express the virtual normal bundle Euler class $e(N_\Gamma^{vir})$ explicitly in terms of graph data.

Factorization Formula

The pullback of the inverse Euler class to the moduli space $\mathcal{M}_\Gamma$ factorizes into vertex and edge contributions:

$$\tau_\Gamma^* \left(\frac{1}{e(N_\Gamma^{vir})} \right) = \prod_{v \in V} \frac{1}{\tilde{N}_\Gamma(v)} \cdot \prod_{e \in E} \frac{1}{\tilde{N}_\Gamma(e)}$$

- **Vertex terms:** lie in $H_T^*(\overline{\mathcal{M}}_{0,\mathrm{val}(v)}) \otimes R^*$ (cohomology of the moduli space factor).
- **Edge terms:** lie in R^* (scalars in the equivariant ring).

Vertex Contributions I: Stable Vertices

A vertex v is **stable** if $\mathrm{val}(v) \geq 3$ (since $g = 0$).

- Let $e_1, \ldots, e_k$ be incident edges connecting v to $v_1, \ldots, v_k$.
- Let ψ_i be the psi-class on $\overline{\mathcal{M}}_{0,\mathrm{val}(v)}$ corresponding to the i-th branch.

Contribution ($g = 0$)

Note: Hodge classes vanish in genus 0.

$$\frac{1}{\tilde{N}_\Gamma(v)} = \frac{1}{e(T_{p_{j(v)}})} \prod_{i=1}^{\mathrm{val}(v)} \frac{1}{\frac{\chi_{j(v)} - \chi_{j(v_i)}}{\delta(e_i)} - \psi_i}$$

Interpretation: This term mixes the tangent space of the target $\mathbb{P}^m$ with the intersection theory (ψ-classes) of the moduli of curves.

Vertex Contributions II: Unstable Vertices

If a vertex is unstable ($\mathrm{val}(v) < 3$), the moduli space is a point. The contribution is purely equivariant weights.

1. **Valence 2 (Unmarked):** Two incident edges e_1, e_2.

$$\frac{1}{\tilde{N}_\Gamma(v)} = \frac{1}{e(T_{p_{j(v)}})} \cdot \frac{1}{\frac{\chi_{j(v)} - \chi_{j(v_1)}}{\delta(e_1)} + \frac{\chi_{j(v)} - \chi_{j(v_2)}}{\delta(e_2)}}$$

2. **Valence 2 (1-Marked):** One edge e, one marking.

$$\frac{1}{\tilde{N}_\Gamma(v)} = \frac{1}{e(T_{p_{j(v)}})}$$

3. **Valence 1 (Unmarked):** Unique incident edge e.

$$\frac{1}{\tilde{N}_\Gamma(v)} = \frac{1}{e(T_{p_{j(v)}})} \cdot \left(\frac{\chi_{j(v)} - \chi_{j(v')}}{\delta(e)} \right)$$

Edge Contributions

Let $e \in E$ be an edge of degree $\delta(e)$ connecting vertices v and v'.
The edge contribution arises from the deformation theory of the map to the invariant line $L_{jj'}$:

Formula

$$\frac{1}{\tilde{N}_\Gamma(e)} = \frac{(-1)^{\delta(e)}\delta(e)^{2\delta(e)}}{(\delta(e)!)^2(\chi_{j(v)} - \chi_{j(v')})^{2\delta(e)}} \cdot \frac{e(T_{p_{j(v)}})e(T_{p_{j(v')}})}{\mathcal{P}_e}$$

Where the product term $\mathcal{P}_e$ accounts for normal directions:

$$\mathcal{P}_e = \prod_{k \notin \{j(v), j(v')\}} \prod_{i=0}^{\delta(e)} \left(\frac{(\delta(e) - i)\chi_{j(v)} + i\chi_{j(v')}}{\delta(e)} - \chi_k \right)$$

The Master Integration Formula

Combining these pieces, we obtain the residue formula for equivariant integrals. Let $\xi \in H_T^*(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d))$ be an equivariant class.

Theorem (Localization Formula)

$$\int_{[\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m,d)]^{vir}} \xi = \sum_{\Gamma \in \mathcal{G}_{0,n}} \frac{1}{|A_\Gamma|} \int_{\mathcal{M}_\Gamma} \frac{\tau_\Gamma^*(\xi)}{\tilde{N}_\Gamma}$$

Key Takeaway: The right-hand side involves integrals over products of $\overline{\mathcal{M}}_{0,k}$ (points, lines, etc.), reducing the complex Gromov-Witten problem to combinatorics and classical intersection numbers.