OpenAI's disproof of Kuznetsov's conjecture

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The conjecture

Let $X \subset \mathbb{P}^5_{\mathbb{C}}$ be a smooth cubic fourfold. Recall

$$D^b(X) = \langle \mathrm{Ku}(X), \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle,$$

where $\mathrm{Ku}(X)$ is the right orthogonal to the three line bundles.

Kuznetsov's conjecture:

$$X \text{ rational} \iff \mathrm{Ku}(X) \simeq D^b(S)$$

for some smooth projective *untwisted* K3 surface S.

The category $\mathrm{Ku}(X)$ is always K3-like, but need not be $D^b(S)$.

Kuznetsov, *Derived categories of cubic fourfolds* (2010).

Previous developments

Following Katzarkov–Kontsevich–Pantev–Yu, Guéré adapted quantum-cohomological methods to prove:

Theorem (Guéré)

If a smooth complex cubic fourfold X is rational, then there is a projective K3 surface S and an isomorphism of rational Hodge structures

$$H^4(X,\mathbb{Q})_{\mathrm{prim}} \;\cong\; H^2(S,\mathbb{Q})(-1).$$

A rational Hodge isomorphism does not retain the *integral* lattice or its discriminant.

Guéré, *On the irrationality of cubic fourfolds* (2026), Theorem 63.

Special cubic fourfolds and Hassett divisors

For a very general cubic fourfold, $H^{2,2}(X,\mathbb{Z}) = \mathbb{Z}h^2$. A cubic X is **special** if it admits a primitive rank-two algebraic lattice

$$K \subset H^{2,2}(X,\mathbb{Z}), \qquad h^2 \in K, \qquad d = \det K.$$

Its moduli divisor is $\mathcal{C}_d$.

Admissibility is the numerical condition

$$d > 6, \quad d \equiv 0,2 \pmod 6, \quad 4 \nmid d, \quad 9 \nmid d,$$
$$p \nmid d \quad \text{for odd primes } p \equiv 2 \pmod 3.$$

For such d, there is an associated polarized K3 (S, L):

$$K^\perp \simeq L^\perp(-1), \qquad L^2 = d.$$

Hassett (2000). Examples: $d = 14, 26, 38, 42, 62, \ldots$

From Hodge structures to categories

The classification of *geometric* Kuznetsov components is an integral lattice statement:

$$\boxed{\begin{array}{c}\mathrm{Ku}(X) \simeq D^b(S) \text{ for some projective K3 } S \\ \Updownarrow \\ X \in \mathcal{C}_d \text{ for some admissible } d.\end{array}}$$

- Guéré's theorem gives a rational Hodge-theoretic condition; admissibility requires integral information.
- The OpenAI preprint contradicts the *sufficiency* direction of Kuznetsov's conjecture, not its necessity.

Addington–Thomas (2014); Huybrechts (2017); Bayer–Lahoz–Macrì–Nuer–Perry–Stellari (2019).

Where the conjecture came from: Pfaffian cubics

Take $V = \mathbb{C}^6$ and a general six-plane $W \subset \Lambda^2 V^\vee$. The Pfaffian construction gives

$$X = \mathbb{P}(W) \cap \mathrm{Pf}, \qquad S = \mathrm{Gr}(2, V) \cap \mathbb{P}(W^\perp).$$

Here X is a cubic fourfold and S a K3 surface of degree 14. The classical results give

$$X \text{ rational}, \qquad F(X) \cong S^{[2]}, \qquad \mathrm{Ku}(X) \simeq D^b(S).$$

This supplied strong evidence for the conjecture.

Beauville–Donagi (1985); Kuznetsov (2006, 2010).

OpenAI's counterexamples

Theorem (OpenAI, September 2026 preprint)

There exists $d_0 > 0$ such that, for every ***admissible*** *$d > d_0$, a* ***very general*** *$X \in \mathcal{C}_d$ satisfies*

$$\mathrm{Ku}(X) \simeq D^b(S) \qquad \textit{and} \qquad X \textit{ is irrational.}$$

The bound d_0 is *ineffective*.

The question is therefore: **how does the proof separate rationality from having a geometric K3 category?**

OpenAI, *Irrational cubic fourfolds with geometric K3 categories* (2026), Theorem 1.1.

Basic idea

Assume that there exists a birational map

$$f : \mathbb{P}^4 \dashrightarrow X.$$

Associate an integer $c_u(f)$ to this hypothetical map and compute it in two different ways:

$$\text{weak factorization} + \text{GW theory} \implies \boxed{c_u(f) = 1},$$

$$\text{Sarkisov program} + \text{boundedness} \implies \boxed{c_u(f) = 0}.$$

The two computations use *the same* divisorial invariant.

Basic setup: transcendental lattices

Let M be a rationally connected smooth projective fourfold. Put

$$L_M = H^4(M,\mathbb{Z})/\mathrm{tors}, \qquad T_M = L_M \cap H^4(M,\mathbb{Q})^{\perp}_{\mathrm{Hdg}},$$

with cup-product form q_M.

For a smooth projective surface Z, define $T_Z \subset H^2(Z,\mathbb{Z})/\mathrm{tors}$ analogously. The restricted forms are nondegenerate over $\mathbb{Q}$.

If $\widetilde{M} = \mathrm{Bl}_Z M$, the classical blowup formula gives

$$\boxed{(T_{\widetilde{M}}, q_{\widetilde{M}}) \cong (T_M, q_M) \perp (T_Z(-1), -q_Z).}$$

The subtle point: *which whole integral summand may a later blowdown remove?*

Integral cohomological blowup formula; OpenAI (2026), §3.

The divisorial invariant

Fix the Picard-rank-one K3 surfaces S for which

$$(T_S(-1), -q_S) \cong (T_X, q_X)$$

as *integral Hodge lattices*.

For a smooth surface Z, set $u(Z) = 1$ if Z is birational to one of these surfaces, and $u(Z) = 0$ otherwise.

Along a weak factorization $f : \mathbb{P}^4 \dashrightarrow X$,

$$c_u(f) = \sum_{\text{surface blowups}} u(Z) - \sum_{\text{surface blowdowns}} u(Z).$$

The signed count is independent of the chosen factorization: its intrinsic definition uses divisorial valuations.

OpenAI (2026), §2; Lin–Shinder's divisorial birational cocycle.

What is new? Gromov–Witten operators

Fix an effective curve class β, descendant powers, and rational diagonal Hodge insertions γ_i. For $u, v \in T_{M,\mathbb{Q}}$, set

$$B(u,v) = \big\langle \tau_a(u), \tau_b(v), \\ \tau_{a_1}(\gamma_1), \dots, \tau_{a_n}(\gamma_n) \big\rangle_{0,\beta}^{M}.$$

Since $q_M|_{T_{M,\mathbb{Q}}}$ is nondegenerate, this yields a Hodge endomorphism

$$B(u,v) = q_M(A_B u, v), \qquad A_B \in \mathrm{End}_{\mathrm{Hdg}}(T_{M,\mathbb{Q}}).$$

Let G_M be the unital $\mathbb{Q}$-algebra generated by all such A_B.

OpenAI (2026), Definition 3.2.

What is new? Integral blocks are detected

Operator blowup theorem (OpenAI). For a smooth surface blowup $\widetilde{M} = \mathrm{Bl}_Z M$,

$$\boxed{G_{\widetilde{M}} \cong G_M \times \mathbb{Q}\,\mathrm{id}_{T_Z}.}$$

(The extra factor is omitted if $T_Z = 0$.)

Along a weak factorization starting at $\mathbb{P}^4$,

$$T_M = V_1^{\mathbb{Z}} \perp \cdots \perp V_m^{\mathbb{Z}},$$
$$G_M = \mathbb{Q}P_1 \times \cdots \times \mathbb{Q}P_m.$$

The idempotent P_i detects the *whole integral block* $V_i^{\mathbb{Z}}$. Thus a subsequent blowdown must remove one such block.

OpenAI (2026), Theorem 3.3 and Lemma 3.8.

The first computation: $c_u(f) = 1$

For a **very general** $X \in \mathcal{C}_d$,

$$\operatorname{rk} T_X = 21, \qquad |\det T_X| = d, \qquad \operatorname{End}_{\mathrm{Hdg}}(T_{X,\mathbb{Q}}) = \mathbb{Q}.$$

If $f : \mathbb{P}^4 \dashrightarrow X$ were birational, its surviving transcendental blocks would satisfy

$$1 = h^{3,1}(X) = \sum_{\alpha} h^{2,0}(Z_\alpha).$$

Hence there is *exactly one* nonzero block. Surface classification identifies it with a K3 surface S such that

$$\operatorname{Pic}(S) = \mathbb{Z}L, \qquad L^2 = d, \qquad \operatorname{Aut}(S) = \{1\}.$$

$$\boxed{c_u(f) = 1}$$

OpenAI (2026), Lemma 3.9 and Proposition 3.10.

The second computation: $c_u(f) = 0$

The Sarkisov program factors f into links between *Mori fibre spaces*:

$$\mathbb{P}^4 \dashrightarrow V_1/B_1 \dashrightarrow \cdots \dashrightarrow V_r/B_r \dashrightarrow X.$$

OpenAI's link corrections. For $d \gg 0$, there are integer-valued corrections $\Theta = v + s$ on the nodes such that every link ℓ satisfies

$$\boxed{c_u(\ell) = \Theta(\mathrm{target}) - \Theta(\mathrm{source}).}$$

The terms telescope. Both endpoints are Mori fibre spaces over pt, hence $\Theta(\mathbb{P}^4) = \Theta(X) = 0$:

$$\boxed{c_u(f) = 0.}$$

OpenAI (2026), §§5–6, especially Theorem 6.6. The bound on d is ineffective.

The contradiction

For the same birational map $f : \mathbb{P}^4 \dashrightarrow X$ we have

$$\underbrace{c_u(f) = 1}_{\substack{\text{GW operators} \\ \text{and weak factorization}}} \qquad \text{and} \qquad \underbrace{c_u(f) = 0}_{\substack{\text{Sarkisov links} \\ \text{and boundedness}}} .$$

Therefore X cannot be rational.

The proof's novel point is the control of *integral* transcendental lattice blocks throughout a factorization.

OpenAI (2026), Theorem 1.1 and Proposition 7.1.

Final remarks

- From the GW point of view, the central observation is closely related to Iritani's blowup formula. **J. Guéré will explain this connection in a forthcoming preprint, expected next Monday (2 November 2026).** Guéré also informed us he have conclude the other direction of Kuznetsov conjecture.
- It remains to see which further irrationality results can be obtained by keeping track of these integral lattice summands. Guéré presented proof of irrationality for very general Verra fourfold.

Thank you!