F-bundles, Hodge atoms, and applications

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October 17, 2025

Outline

1. **nc**-Hodge structures
2. F-bundles

Recall

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- an *algebraic vector bundle* H of rank r over a scheme X is a *locally free sheaf* of $\mathcal{O}_X$-modules of rank r, i.e., for every $x \in X$ there is $X \supset U \ni x$ such that $H|_U \cong \mathcal{O}_X(U)^{\oplus r}$.

Recall

Let X be a smooth scheme over a field k. Let $H \to X$ be an algebraic vector bundle over X. Let $\Omega^1_{X/k}$ be the sheaf of $\mathcal{O}_X$-module of Kähler differentials (1-forms) on X over k.

Definition 1

A *connection* on $\mathcal{E}$ is a k-linear morphism of sheaves:

$$\nabla : H \to H \otimes_{\mathcal{O}_X} \Omega^1_{X/k}$$

satisfying Leibniz rule:

$$\nabla(f.s) = s \otimes \mathrm{d}f + f \cdot \nabla s$$

for any open set $U \subset X$ and local sections $f \in \mathcal{O}_X(U),\ s \in H|_U$.

Recall

Let X be a smooth scheme over a field k, H an algebraic vector bundle (locally free sheaf) on X, and $D \subset X$ a divisor.

Definition 2

A *meromorphic connection* ∇ on H with poles along D is a k-linear morphism of sheaves:

$$\nabla : H \to H \otimes_{\mathcal{O}_X} \Omega^1_{X/k}(*D)$$

This operator satisfies the Leibniz rule:

$$\nabla(f \cdot s) = s \otimes \mathrm{d}f + f \cdot \nabla s$$

for local regular functions $f \in \mathcal{O}_X$ and local sections $s \in H$.

Note: $\Omega^1_{X/k}(*D)$ denotes the sheaf of $\Omega^1_{X/k}$ forms with allowed poles (meromorphic singularities) only along the divisor D.

nc-Hodge structures

Definition 3

A *pure* $\mathbb{Q}$-**nc**-*Hodge structure is a triple* $(H, \nabla, \mathcal{E}_B)$, where

- $H \to \mathbb{C}$ is a $\mathbb{Z}/2$-graded algebraic $\mathbb{C}$-vector bundle

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- $H \to \mathbb{C}$ is a $\mathbb{Z}/2$-graded algebraic $\mathbb{C}$-vector bundle
- $\mathcal{E}_B \subset H_{\mathbb{C}\setminus\{0\}}$ is a $\mathbb{Q}$-local system generationg H, i.e.

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- a meromorphic connection on H with pole of order ≤ 2 at $0 \in \mathbb{C}$.

Moreover: $\mathcal{E}_B$ is ∇-flat, compatible with Stokes data, the vector bundle $\widehat{H}$ over $\mathbb{P}^1$ obtained by gluing H with its complex conjugate is holomorphically trivial.

nc-Hodge structures: meaning

- $\mathcal{E}_B$ is ∇-flat: horizontal sections ($\nabla s = 0$) are the sections of the local system $\mathcal{E}_B$.

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- Stokes data (or the Stokes phenomenon) describes how the local solutions of a differential equation near an irregular singular point (like a pole of order 2) transition across certain rays in the complex plane. This compatibility ensures that the bundle H is analytically consistent with the given singular connection ∇.
- $\widehat{H} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus 2r}$. The Birkhoff–Grothendieck Theorem states that every algebraic vector bundle over $\mathbb{P}^1$ is a direct sum of line bundles: $H \cong \bigoplus \mathcal{O}_{\mathbb{P}^1}(k_i)$. The condition of being trivial means all these degrees k_i must be 0.

From Classical to NC

Recall: A classical $\mathbb{Q}$-pure Hodge structure of weight w:

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- A finite-dimensional $\mathbb{Q}$-vector space $H_{\mathbb{Q}}$.
- A decreasing **Hodge filtration** $F^\bullet$ on its complexification $H_{\mathbb{C}} = H_{\mathbb{Q}} \otimes \mathbb{C}$:

$$H_{\mathbb{C}} = F^0 \supset F^1 \supset \cdots \supset F^w \supset \{0\}$$

such that $H_{\mathbb{C}} = F^p \oplus \overline{F^{w-p+1}}$.

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The Task

Construct the triple $(H, \nabla, \mathcal{E}_B)$ using only the data $(H_{\mathbb{Q}}, F^\bullet)$.

Step 1: The Bundle and Local System

This part is straightforward. We define the vector bundle H and the local system $\mathcal{E}_B$ from our classical data.

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- **The Local System** $\mathcal{E}_B$: We use the rational structure $H_{\mathbb{Q}}$ to define a $\mathbb{Q}$-local system on the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Take the *constant sheaf* with stalk $H_{\mathbb{Q}}$:

$$\mathcal{E}_B := (H_{\mathbb{Q}})_{\mathbb{C}^*}$$

  This satisfies the condition $H|_{\mathbb{C}^*} \cong \mathcal{E}_B \otimes \mathcal{O}_{\mathbb{C}^*}$ since $H_{\mathbb{Q}} \otimes \mathbb{C} \cong H_{\mathbb{C}}$.

Step 2a: The Nilpotent Operator N

We define a linear operator $N : H_{\mathbb{C}} \to H_{\mathbb{C}}$ with the property:

$$N(H^{p,q}) \subseteq H^{p-1,q+1}$$

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Since $H^{p,q} = 0$ for $p < 0$, applying N repeatedly will eventually yield zero.
Thus, N is a **nilpotent operator**.

The Contraction Operator Λ: Geometric Definition

The nilpotent operator N used in the **nc**-Hodge construction is rooted in the geometry of a Kähler manifold X, where it is related to the operator Λ.
Context: Let X be a compact Kähler manifold, and $[\omega] \in H^2(X, \mathbb{R})$ the class of the Kähler form.

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- **Lefschetz Operator (L):** This operator *raises* the degree by cup product with $[\omega]$.

$$L : H^k(X) \to H^{k+2}(X), \quad L(\alpha) = [\omega] \cup \alpha$$

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- **Contraction Operator (Λ):** This operator *lowers* the degree. It is the **adjoint** of L with respect to a specific bilinear form.

$$\Lambda : H^{k+2}(X) \to H^k(X)$$

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Defining Λ: The Closed Expression

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The Polarization Form (Q)

Q is a non-degenerate, $\mathbb{Q}$-bilinear form on $H^*(X, \mathbb{C})$ (the Hodge-Riemann bilinear relation). For $H^k(X)$, the form Q_k is defined by:

$$Q_k(\alpha, \beta) = (\sqrt{-1})^{k^2} \int_X \alpha \cup \beta \cup [\omega]^{n-k}$$

where $n = \dim_{\mathbb{C}} X$.

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Λ is uniquely defined by the condition that it is the adjoint of L with respect to Q:

$$Q_{k+2}(L(\alpha), \beta) = Q_k(\alpha, \Lambda(\beta))$$

for all $\alpha \in H^k(X, \mathbb{C})$ and $\beta \in H^{k+2}(X, \mathbb{C})$.

The $\mathfrak{sl}_2$ Commutation Relations

The operators L and Λ, along with the Weight Operator H, form a representation of the $\mathfrak{sl}_2$ Lie algebra on the cohomology $H^*(X, \mathbb{C})$.

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- **Commutation Relation 3: H and Λ**
$$[H, \Lambda] = H\Lambda - \Lambda H = -2\Lambda$$

Step 2b: Connection Construction ∇

Recall: The nilpotent operator N is derived from the Hodge structure and satisfies the filtration-lowering property:

$$N(F^p) \subset F^{p-1}$$

- **Goal**: Define ∇ on the trivial bundle $H = H_{\mathbb{C}} \times \mathbb{C}$ such that its analytic properties at $z = 0$ encode the Hodge filtration $F^\bullet$.

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- **Construction**: We use N to define a connection with an **irregular singularity** (pole of order 2) at the origin $z = 0$:

$$\nabla := d - \frac{N}{z^2} dz$$

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The Differential Equation

Setting $\nabla\psi = 0$ yields the following ordinary differential equation (ODE) for the sections $\psi(z)$:

$$d\psi - \frac{N}{z^2} dz \cdot \psi = 0$$

Which can be written as:

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Note: Because the coefficient $\frac{N}{z^2}$ has a pole of order 2, $z = 0$ is an irregular singularity.

Step 2d: The Formal Solution and Series

The differential equation has a formal solution derived by integration.

Formal Solution

The solution is given by integrating the connection one-form:

$$\psi(z) = \exp\left(\int \frac{N}{z^2} dz\right) \cdot v_0 = \exp\left(-\frac{N}{z}\right) v_0, \quad v_0 \in H_{\mathbb{C}}$$

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Since N is nilpotent ($N^k = 0$), the exponential is a finite Taylor series in $1/z$:

$$\exp\left(-\frac{N}{z}\right) = \mathbf{Id} - \frac{N}{z} + \frac{N^2}{2!z^2} - \cdots + \frac{(-1)^{k-1} N^{k-1}}{(k-1)! z^{k-1}}$$

Encoding Result I: The Component-wise Action

Applying the series expansion to an initial vector $v \in H_{\mathbb{C}}$ shows the mechanism for encoding the filtration $F^{\bullet}$.

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Action of N on a Filtered Vector

If we choose a base vector $v \in F^p \subset H_{\mathbb{C}}$, the solution $\psi(z)$ near $z = 0$ is:

$$\begin{aligned}\psi(z) &= \left(\mathbf{Id} - \frac{N}{z} + \frac{N^2}{2!z^2} - \dots\right) v \\ &= \underbrace{v}_{\in F^p} + \underbrace{\left(-\frac{1}{z}\right) N(v)}_{\in F^{p-1}} + \underbrace{\left(\frac{1}{2!z^2}\right) N^2(v)}_{\in F^{p-2}} + \dots\end{aligned}$$

Each coefficient in the asymptotic series belongs to a successively lower filtration level due to the property $N(F^p) \subset F^{p-1}$.

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Encoding Result II: The Filtration Recovery

Explicit Encoding

The Hodge filtration $F^\bullet$ is precisely the data encoded in the coefficients (the powers of N) of the asymptotic expansion of the solution $\psi(z)$ near $z = 0$.

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- ... and so on.

The **Stokes data** (how these asymptotic solutions are glued globally) is what fully recovers the classical Hodge filtration $F^\bullet$.

Conclusion: The Full Picture

Let us assemble the pieces. Starting from a classical Hodge structure $(H_{\mathbb{Q}}, F^{\bullet})$, we have constructed a triple $(H, \nabla, \mathcal{E}_B)$.

- $H = H_{\mathbb{C}} \times \mathbb{C}$
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Result

This triple $(H, \nabla, \mathcal{E}_B)$ is a pure $\mathbb{Q}$-**nc**-Hodge structure.

Why does this work?

- The local solutions to $\nabla(s) = 0$ near $z = 0$ have asymptotic expansions involving $\exp(N/z)$.
- The nilpotent operator N was built from the Hodge filtration $F^\bullet$.
- The **Stokes data** of the connection ∇ (how these solutions are glued across different sectors around $z = 0$) precisely recovers the filtration $F^\bullet$.

Quantum connection

For a compact symplectic manifold (X, ω) of dimension $2d$, the associated **nc**-Hodge structure has a specific connection ∇.

Definition 4 (Quantum Connection)

The ∇ part of the **nc**-Hodge structure is the **quantum connection**:

$$\nabla_{\partial_u} = \partial_u - u^{-2}\left(c_1(X) *_q (\bullet)\right) + u^{-1}\mathbf{Gr}$$

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- **Bundle**: $H = H^\bullet(X, \mathbb{C}) \otimes \mathcal{O}$.
- **Grading Operator** (**Gr**): $\mathbf{Gr} = (k - d)/2\,\mathrm{id}$ on $H^k(X, \mathbb{C})$.
- **Quantum Product** ($*_q$): $H^\bullet(X, \mathbb{C})^{\otimes 2} \to H^\bullet(X, \mathbb{C}) \otimes \mathbb{C}_q$.

Definition of an F-bundle

Let $\Bbbk$ be a non-archimedean field.

Definition 5 (F-bundle $(\mathcal{B}, \mathscr{H}, \nabla)$)

An **F-bundle** is the data $(\mathcal{B}, \mathscr{H}, \nabla)$ where:

- $\mathcal{B}$ is a supermanifold over $\Bbbk$.
- $\mathscr{H}$ a super vector bundle over $\mathcal{B} \times \mathbb{D}$.
- ∇ is a meromorphic connection on $\mathscr{H}$ with poles at most at $u = 0$.

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Regularity Conditions

The connection ∇ must satisfy the following:

- $\nabla^2 = 0$.
- $\nabla_{u^2\partial_u}$ is **regular** (no poles).
- $\nabla_{u\xi}$ is **regular**, for any $\xi \in T\mathcal{B}$.

Residual Parts of an F-bundle

An F-bundle $(\mathcal{B}, \mathscr{H}, \nabla)$ induces four key residual structures:

- A vector bundle $\mathscr{H}_{\mathbf{o}}$ on $\mathcal{B}$.
- A relative flat u-connection $(\mathscr{H}, \boldsymbol{\delta})$ on $\mathcal{B} \times \mathbb{D} \to \mathbb{D}$.
- A regular **Higgs field** $\theta : \mathscr{H}_o \to \mathscr{H}_o \otimes \Omega^1_{\mathcal{B}}$.
- A regular endomorphism $\kappa : \mathscr{H}_o \to \mathscr{H}_o$.

Definitions:

$\mathscr{H}_o$ The restriction $\mathscr{H}_{|\mathcal{B}\times\{o\}}$.

$\boldsymbol{\delta}$ Projection of $u\nabla$ to forms along the fibers of $\mathcal{B} \times \mathbb{D} \to \mathcal{B}$.

θ The restriction of $\boldsymbol{\delta}$ to $u = 0$. (Must satisfy $\theta \wedge \theta = 0$).

κ The restriction of $\nabla_{u^2\partial_u}$ to $u = 0$. (κ commutes with θ).

Maximality and the F-manifold

The induced Higgs field θ allows us to realize vector fields on $\mathcal{B}$ as a commuting family of $\mathcal{O}_{\mathcal{B}}$-linear operators:

$$\theta \lrcorner (-) : T_{\mathcal{B}} \to \mathsf{End}(\mathscr{H}_o)$$

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An F-bundle $(\mathcal{B}, \mathscr{H}, \nabla)$ is **maximal at** $b \in \mathcal{B}$ if there exists an even vector $h \in (\mathscr{H}_o)_b$ such that the map

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If $(\mathcal{B}, \mathscr{H}, \nabla)$ is maximal, then:

- $T_{\mathcal{B}}$ inherits a unital, commutative, associative algebra structure.
- This structure embeds $T_{\mathcal{B}}$ into $\mathrm{End}(\mathscr{H}_o)$.
- Consequently, $\mathcal{B}$ is a formal F-**manifold** (in the sense of Hertling-Manin).

Euler Vector Field and Decomposition

A maximal F-bundle $\mathcal{B}$ is equipped with a unique vector field $\mathsf{Eu} \in \Gamma(\mathcal{B}, T_{\mathcal{B}})$ called the **Euler vector field** such that:

$$\theta \lrcorner \mathsf{Eu} = \kappa$$

The powers of Eu (with respect to the product on $T_{\mathcal{B}}$) generate an action of the Witt Lie algebra on $\mathcal{B}$.

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Theorem 2.1 (Katzarkov-Kontsevich-P-Yu'21, Yu-Zhang'24)

Let $b \in \mathcal{B}$ be a closed point. The decomposition of $\mathscr{H}_{(b,o)} = \oplus_\lambda \mathscr{H}^\lambda_{b,0}$ into generalized eigenspaces for κ_b induces a canonical decomposition of the germ of $(\mathcal{B}, \mathscr{H}, \nabla)$ into a product of maximal F-bundles:

$$\bigoplus_{\lambda \in \mathrm{spectrum}(\kappa_b)} \left(\mathcal{B}^\lambda, \mathscr{H}^\lambda, \nabla^\lambda\right).$$

The Main Example: A-model F-bundle

Universal Source of F-bundles

The most important examples of maximal F-bundles arise from **variations of nc-Hodge structures of geometric origin**.
Specifically, those corresponding to smooth proper families of **nc**-spaces satisfying the Hodge-to-de Rham degeneration property.

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Our Focus: We are interested in the structure that goes by the name of the **A-model F-bundle**.

The A-model F-bundle: Generating Series $\mathscr{F}_0$

Let (X, ω) be a compact symplectic manifold with $[\omega] \in \mathrm{im}[H^2(X, \mathbb{Z}) \to H^2_{dR}(X)]$.

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The Generating Series $\mathscr{F}_0$

These invariants are packaged into the genus zero generating series:

$$\mathscr{F}_0 = \sum_{\beta \in H_2(X,\mathbb{Z})} q^{\int_\beta [w]} \sum_{n \geq 0} \sum_{\alpha_1, \ldots, \alpha_n} \mathrm{sign}_{n,\beta} \int_{I_{n,\beta}} (\Delta_{\alpha_1} \boxtimes \cdots \boxtimes \Delta_{\alpha_n}) \cdot t_{\alpha_1} \cdots t_{\alpha_n},$$

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Variables and Basis:

- $\{\Delta_\alpha\}$: Graded basis of the cohomology $H^\bullet(X, \mathbb{Q})$.

Coordinates:

- $\{t_\alpha\}$: Supervariables dual to the basis $\{\Delta_\alpha\}$.
- q: Formal variable for curve classes (via $q^{\int_\beta [w]}$).

The A-model F-bundle: Structures on the Base $\mathcal{B}$

Let $\Bbbk = \overline{\mathbb{Q}}((\boldsymbol{y}))$ and let ${}^{X}\mathcal{B}$ be an open ball in the analytic space $H^\bullet(X, \mathbb{Q}) \otimes \Bbbk$.

The Analytic Space $\mathcal{B}$

- $\mathscr{F}_0$ can be viewed as a function on the base space ${}^{X}\mathcal{B}$.
- The Poincaré pairing η on $H^\bullet(X)$ gives a flat metric on ${}^{X}\mathcal{B}$:

$$\sum \eta_{\alpha\alpha'} dt_\alpha dt_{\alpha'}$$

The Quantum Product $*$ At each point of ${}^{X}\mathcal{B}$, the generating function $\mathscr{F}_0$ induces the fundamental algebraic structure: the **quantum product** $*$ on $H^\bullet(X)$.

$$\Delta_{\alpha_1} * \Delta_{\alpha_2} = \sum_{\alpha,\alpha_3} \frac{\partial^3 \mathscr{F}_0}{\partial t_{\alpha_1} \partial t_{\alpha_2} \partial t_{\alpha_3}} \cdot \left(\eta^{-1}\right)_{\alpha\alpha_3} \Delta_\alpha.$$

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- The $\mathbb{Q}$-local system $\mathcal{E}_B$ is naturally included.
- The Vector Bundle ${}^{\mathbf{X}}\mathscr{H}$: This is the trivial bundle on $\mathcal{B} \times \mathbb{D}$ with fiber $H^\bullet(X)$.

The A-model F-bundle: The Quantum Connection ${}^{X}\nabla$

The *A*-**model** *F* **bundle** associated with X is the triple $({}^{X}\mathcal{B}, {}^{X}\mathscr{H}, {}^{X}\nabla)$, where ${}^{X}\nabla$ is the **quantum connection**.

The connection has two components: one along the disk coordinate u and one along the base coordinates t_α.

$$\left| \begin{array}{l} {}^{X}\nabla_{\partial_u} = \partial_u + u^{-2}\mathsf{K} + u^{-1}\mathsf{Gr}, \\ {}^{X}\nabla_{\partial_{t_\alpha}} = \partial_{t_\alpha} + u^{-1}\mathsf{A}_\alpha. \end{array} \right.$$

Components of the Connection ${}^{X}\nabla$

A_α: The Higgs field component, defined by multiplication:

$$\mathsf{A}_\alpha = \Delta_\alpha * (-)$$

Gr: The grading operator (Euler vector field) on $H^\bullet(X)$:

$$\mathsf{Gr} = \bigoplus_{i=0}^{2d} \frac{i-d}{2} \, \mathrm{id}_{H^i(X)}$$

K: The $\mathbf{u}^{-2}$ residue component, derived from the first Chern class $c_1(X)$:

$$\mathsf{K} = \left[-c_1(X) + \sum_{a, \deg \Delta_\alpha \neq 0} \frac{\deg \Delta_\alpha - 2}{2} t_\alpha \Delta_\alpha \right] * (-)$$

The Blow-Up Theorem: F-bundle Decomposition (Iritani)

Geometric Setup (Blow-up)

Suppose X is a smooth projective manifold over $\mathbb{C}$.

- Let $Y \subset X$ be a smooth subvariety with $\operatorname{codim}_Y X = m \geq 2$.
- Let $\widetilde{X} = \mathrm{Bl}_Y(X)$ be the manifold obtained by blowing up X along Y.

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Theorem 7 (Iritani, 2023)

There exist specific limiting points in the base spaces ${}^{\widetilde{X}}\mathcal{B}$ and ${}^{X}\mathcal{B} \times {}^{Y}\mathcal{B}^{\times(m-1)}$ such that we have an isomorphism of germs of F-bundles:

$$\left({}^{\widetilde{X}}\mathcal{B}, {}^{\widetilde{X}}\mathscr{H}, {}^{\widetilde{X}}\nabla\right) \quad \cong \quad \left({}^{X}\mathcal{B}, {}^{X}\mathscr{H}, {}^{X}\nabla\right) \times \left({}^{Y}\mathcal{B}, {}^{Y}\mathscr{H}, {}^{Y}\nabla\right)^{\times(m-1)}$$

This isomorphism holds for the germs taken at the respective limiting points.

The Blow-Up Theorem: F-bundle Decomposition

Interpretation: F-Bundle Factorization

The A-model F-bundle associated with the blow-up $\widetilde{X}$ decomposes as a product of the F-bundles associated with the original manifold X and the center of the blow-up Y (appearing $m-1$ times, where m is the codimension).