$\star$-Diagrams: Explicit Constructions, Geometry, and a Symplectic Obstruction

Talk notes

Attribution convention. Statements marked **[K]** are known (Speranca; Speranca–Cavenaghi; Milnor; Kervaire–Milnor; Brieskorn; Hirzebruch–Mayer). Statements marked **[N]** are new: the endomorphism A, the Theorem of Part IV, and the explicit computations. *Say this out loud once, early.*

Board hygiene. Reserve a corner of the board, write this once, never erase:

G acts on M on the LEFT:	$g \cdot x, \quad L_g(x) := g \cdot x$
principal action of P:	$p \bullet s$ (RIGHT)
second action of P:	$r \star p$ (LEFT)
conjugation-equivariance:	$\varphi(g \cdot x) = g\,\varphi(x)\,g^{-1}$
reentrance:	$\hat{\varphi}(x) = \varphi(x) \cdot x$
Milnor:	$e(M_{h,j}) = h + j, \qquad \frac{1}{2}p_1(M_{h,j}) = 2(h - j)$

Every sign error in this talk is a left/right error or a Milnor-convention error.

Part 0. The question [4 min]

Write on the board:

$$S^{2n-1} \qquad \text{and} \qquad \Sigma^{2n-1} = \partial P^{2n}(A_2).$$

> I will describe a machine which takes a single conjugation-equivariant map $t : X \to G$ and produces two manifolds with the same orbit space, the same isotropy data, and — sometimes — different smooth structures. Then I will ask whether that machine can be run symplectically, and prove that it cannot.

Announce the destination so people stay: *an explicit endomorphism A of TM, computed in closed form, whose vanishing is exactly the condition for the construction to be symplectic on cotangent bundles — and which is exactly the condition for the construction to be trivial.*

Part I. The engine [14 min]

I.1 Basic definitions [3 min]

Definition 1 (G-manifold)**.** A smooth left action $G \times M \to M$ of a compact Lie group. Orbits $G \cdot x$, isotropy $G_x = \{g : g \cdot x = x\}$, orbit space M/G. On the principal stratum M_{princ} all isotropy groups are conjugate and M_{princ}/G is a manifold.

Definition 2 (fundamental vector field)**.** For $X \in \mathfrak{g} = \text{Lie}(G)$, $\zeta_X(x) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \exp(tX) \cdot x \in T_xM$. Record the identity used three times below:

$$\zeta_{\text{Ad}_g X}(g \cdot x) = dL_g\, \zeta_X(x), \tag{1}$$

immediate from $\exp(t\,\text{Ad}_g\, X) = g \exp(tX) g^{-1}$.

Definition 3 (principal bundle by transition data)**.** Given an open cover $\{U_i\}$ of M and $\varphi_{ij} : U_i \cap U_j \to G$ with $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$, set

$$P = \Big(\bigsqcup_i U_i \times G\Big)\Big/\sim, \qquad (x,g)_j \sim \big(x, \varphi_{ij}(x)g\big)_i,$$

with principal action $(x,g) \bullet s = (x, gs)$. It is free and $P/\!\bullet = M$.

Definition 4 (Maurer–Cartan forms)**.** $\theta_L|_g(v) = (dL_{g^{-1}})v$ and $\theta_R|_g(v) = (dR_{g^{-1}})v$; in matrix notation $\theta_L = g^{-1}\mathrm{d}g$, $\theta_R = \mathrm{d}g\, g^{-1}$, so $\theta_R = \mathrm{Ad}_g\, \theta_L$. Structure equation $\mathrm{d}\theta_L + \frac{1}{2}[\theta_L, \theta_L] = 0$.

I.2 The $\star$-collection [3 min]

Definition 5 ($\star$-collection, **[K]**)**.** Let M be a G-manifold. A *$\star$-collection* is a *G-invariant* open cover $\{U_i\}$ together with smooth $\varphi_{ij} : U_i \cap U_j \to G$ such that

(i) $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$ (cocycle);
(ii) $\varphi_{ij}(g \cdot x) = g\, \varphi_{ij}(x)\, g^{-1}$ (conjugation-equivariance).

Say both hypotheses out loud. Invariance of the cover is not decorative: it is what makes (ii) type-check, since $g \cdot x$ must lie in the same overlap.

Definition 6 (reentrance map)**.** $\hat{\varphi}_{ij}(x) := \varphi_{ij}(x) \cdot x$.

Remark 7. $\hat{\varphi}_{ij}$ is G-equivariant: $\hat{\varphi}_{ij}(g \cdot x) = \varphi_{ij}(gx) \cdot gx = g\varphi_{ij}(x)g^{-1}g \cdot x = g \cdot \hat{\varphi}_{ij}(x)$. One line; do it on the board.

I.3 The construction theorem [4 min]

Theorem 8 (**[K]**)**.** *Let $\{\varphi_{ij}\}$ be a $\star$-collection on M and $P \to M$ the associated principal bundle. Then, in the chart $U_i \times G$, the formula*

$$r \star (x,g) := \big(r \cdot x,\ rg\big)$$

is well defined on P, and

(1) $\star$ is a free left G-action;
(2) $\star$ commutes with $\bullet$;
(3) $P/\!\star \cong M^\star$, the manifold obtained by regluing the pieces U_i along the reentrance maps;
*(4) hence $M \longleftarrow P \longrightarrow M^\star$ is a G–G biprincipal bundle, a $\star$-*diagram*;*
(5) $M^\star$ carries a residual G-action and the regluing maps are G-equivariant.

Proof, all of it. **Well-definedness.** In chart j, $r \star (x,g)_j = (rx, rg)_j$, glued to

$$\big(rx,\ \varphi_{ij}(rx)\, rg\big)_i = \big(rx,\ r\varphi_{ij}(x)r^{-1}rg\big)_i = \big(rx,\ r\,\varphi_{ij}(x)g\big)_i = r \star \big(x, \varphi_{ij}(x)g\big)_i.$$

This is the only place conjugation-equivariance is used, and it is the reason the definition exists.

Freeness, commutation. In the chart: $r\star(x,g) = (x,g)$ forces $rg = g$; and $r\star\big((x,g)\bullet s\big) = (rx, rgs) = \big(r \star (x,g)\big) \bullet s$.

The quotient. The $\star$-orbit of (x,g) contains $(g^{-1}\cdot x, e)$, so $P|_{U_i}/\!\star \cong U_i$ via $[(x,g)] \mapsto g^{-1}\cdot x$. Put $y = g^{-1}x$, i.e. $x = gy$. Then

$$[(x,g)_j] \mapsto y, \qquad \big[(x, \varphi_{ij}(x)g)_i\big] \mapsto g^{-1}\varphi_{ij}(gy)^{-1}gy = \varphi_{ij}(y)^{-1} \cdot y,$$

using equivariance in the last step. So the regluing maps of $M^\star$ are

$$\Phi_{ij}(y) := \varphi_{ij}(y)^{-1} \cdot y = \hat{\varphi}_{ji}(y). \tag{2}$$

Residual action. $\bullet$ descends: $(x,g) \bullet s \mapsto s^{-1} \cdot y$. □

Remark 9 (ordering, state it once)**.** With the standard bundle cocycle, reentrance maps compose in the *opposite* order, $\hat{\varphi}_{ij} \circ \hat{\varphi}_{jk} = \widehat{\varphi_{jk}\varphi_{ij}}$. The maps (2) fix this: $\Phi_{ij} \circ \Phi_{jk} = \Phi_{ik}$. Note φ_{ij}^{-1} is conjugation-equivariant whenever φ_{ij} is.

I.4 The trivial construction [2 min]

One chart, $\varphi \equiv e$: then $P = M \times G$ with $r \star (x, g) = (r \cdot x, rg)$ and $(M \times G)/\star \cong M$ via $[x, g] \mapsto g^{-1} \cdot x$. Extract three things: the $\star$-action is the *anti-diagonal* one; *every* $\star$-diagram is locally of this form; and all twisting comes from the failure of φ_{ij} to be locally constant — which will reappear as $\tilde{\theta} = a^{-1}\mathrm{d}a$ in Part IV.

I.5 Functoriality [2 min]

Theorem 10 ([**K**] for (1), [**N**] for (2))**.** *(1) If $f : N \to M$ is G-equivariant, then $\{f^{-1}(U_i)\}$ with $\{\varphi_{ij} \circ f\}$ is a $\star$-collection on N, giving $N \leftarrow f^*P \to N^\star$.*

(2) If $E \xrightarrow{\pi_E} M$ is a G-equivariant fibre bundle, then $\{\pi_E^{-1}(U_i)\}$ with $\{\varphi_{ij} \circ \pi_E\}$ is a $\star$-collection on E, giving $E \leftarrow \pi_E^* P \to E^\star$ with $E^\star \to M^\star$ a fibre bundle.

Proof of (2). $\pi_E^{-1}(U_i)$ is G-invariant since π_E is equivariant; the cocycle condition is inherited; and $\varphi_{ij}(\pi_E(g \cdot e)) = \varphi_{ij}(g \cdot \pi_E(e)) = g\varphi_{ij}(\pi_E e)g^{-1}$. □

Part (2) is what makes Part IV possible: *one may take $E = T^*M$.* Flag it now.

I.6 The orbit-space lemma [1 min]

Lemma 11 ([**N**], sharpening a known statement)**.** *$\hat{\varphi}_{ij}(x) \in G \cdot x$. Hence the regluing maps induce the* identity *on $(U_i \cap U_j)/G$, and therefore*

$$M^\star/G \;=\; M/G \qquad \textit{canonically, not merely up to isomorphism.}$$

Part II. Examples [18 min]

II.1 A quaternionic $\star$-collection on S^7 [5 min]

Let $G = S^3 \subset \mathbb{H}$ act on $M = S^7 \subset \mathbb{H} \oplus \mathbb{H}$ by

$$q \cdot (x, y) = (qx\bar{q},\ qy\bar{q}).$$

Cover by the invariant sets $U_1 = \{x \neq 0\}$, $U_2 = \{y \neq 0\}$, and on the overlap put

$$\boxed{a(x, y) = \hat{x}\,\overline{\hat{y}}, \qquad \hat{x} = x/|x|,\ \hat{y} = y/|y|.}$$

Conjugation-equivariance, one line on the board:

$$a\big(qx\bar{q}, qy\bar{q}\big) = (q\hat{x}\bar{q})\overline{(q\hat{y}\bar{q})} = q\hat{x}\bar{q} \cdot q\overline{\hat{y}}\bar{q} = q\,\hat{x}\overline{\hat{y}}\,\bar{q} = q\,a(x, y)\,\bar{q}.$$

For this collection $P \cong \mathrm{Sp}(2)$ and $M^\star \cong \Sigma^7$, an exotic 7-sphere generating $bP_8 = \Theta_7 \cong \mathbb{Z}/28$.

Fixed-point data, if asked: the action fixes the circle $\{(a, b) \in \mathbb{R}^2\} \cap S^7$, with isotropy representation $\rho_1 \oplus \rho_1 \oplus \rho_0$ on $\mathrm{Im}\,\mathbb{H} \oplus \mathrm{Im}\,\mathbb{H} \oplus \mathbb{R}$.

II.2 Plumbing — recalled explicitly [4 min]

Everyone should have this definition in front of them; it is used twice more.

Definition 12 (plumbing along A_2)**.** Let $DTS^n \to S^n$ be the unit disc bundle of the tangent bundle, a $2n$-manifold with boundary. Take two copies, choose $p_i \in S^n$ and local trivialisations

$$DTS^n\big|_{D_i} \;\cong\; D^n \times D^n \qquad \text{(base} \times \text{fibre)}, \qquad p_i \in D_i.$$

Glue the two total spaces by identifying these two copies of $D^n \times D^n$ through the *swap*

$$(u, v) \longmapsto (v, u),$$

exchanging base and fibre coordinates, and smooth the corners. The result is a compact $2n$-manifold with boundary, written $P^{2n}(A_2)$.

Three consequences, all worth writing:

(a) $P^{2n}(A_2)$ deformation retracts onto the wedge $S^n \vee S^n$ of the two zero sections, joined at the plumbing point. So $H_n \cong \mathbb{Z}^2$, generated by $[S_1], [S_2]$, and $P^{2n}(A_2)$ is $(n-1)$-connected.

(b) Intersection numbers: $[S_i] \cdot [S_i] = e(TS^n) = \chi(S^n) = 1 + (-1)^n$, and $[S_1] \cdot [S_2] = 1$ since the two zero sections meet transversally at exactly the plumbing point. Hence

$$Q = \begin{pmatrix} \chi(S^n) & 1 \\ 1 & \chi(S^n) \end{pmatrix}.$$

(c) $P^{2n}(A_2)$ is parallelisable, being built from n-handles on tangent disc bundles of spheres. This is why its boundary lies in bP_{2n}.

II.3 The parity computation [3 min] — do it live

From the long exact sequence of $(P, \partial P)$ together with Poincaré–Lefschetz,

$$H_n(\partial P) \cong \ker Q, \qquad H_{n-1}(\partial P) \cong \operatorname{coker} Q.$$

- n **odd**: $\chi(S^n) = 0$, $Q = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\det Q = -1$. Unimodular, so $\partial P^{2n}(A_2)$ is an $(n-1)$-connected homology $(2n-1)$-sphere, hence a homotopy sphere.
- n **even**: $\chi(S^n) = 2$, $\det Q = 3$, so $H_{n-1}(\partial P) \cong \mathbb{Z}/3$: the boundary is only a $\mathbb{Z}/3$-homology sphere.

Definition 13 (Kervaire sphere)**.** For $n = 2m+1$ odd, $\Sigma^{4m+1} := \partial P^{2n}(A_2)$ is the *Kervaire sphere*. It is a homotopy $(4m+1)$-sphere bounding a parallelisable manifold, and it generates bP_{4m+2}, which is $\mathbb{Z}/2$ or 0.

Insist on the distinction: the Kervaire *manifold* exists for all n; the Kervaire *sphere* only for n odd, i.e. in dimensions $\equiv 1 \bmod 4$.

Remark 14 (Brieskorn description, [**K**])**.** Set $W^{2n-1}(d) = \{z_0^d + z_1^2 + \cdots + z_n^2 = 0\} \cap S^{2n+1}$, a closed $(2n-1)$-manifold. Then $\partial P^{2n}(A_k) \cong W^{2n-1}(k+1)$: for $k = 1$ this is the unit cotangent bundle ST^*S^n, and for $k = 2$ it is the manifold above. For n and d both odd, $W^{2n-1}(d)$ is homeomorphic to S^{2n-1}; it is the standard sphere for $d \equiv \pm 1 \pmod 8$ and the Kervaire sphere for $d \equiv \pm 3 \pmod 8$. As the link of an isolated singularity, Σ^{4m+1} therefore carries a natural contact structure. *Keep this role — as a contact manifold — separate from its role as a smooth base in Part V.*

Theorem 15 ([**K**], black box)**.** $bP_{4m+2} = 0$ *exactly when a framed manifold of Kervaire invariant one exists in dimension* $4m+2$*, i.e. for* $4m+2 \in \{2, 6, 14, 30, 62, 126\}$*. Hence* Σ^{4m+1} *is exotic except in dimensions* $5, 13, 29, 61, 125$*.*

Do not motivate this. Cite Hill–Hopkins–Ravenel and the resolution of dimension 126, and move on.

II.4 A reflection $\star$-collection on S^{2n-1} [4 min]

Let $G = O(n)$ act *biaxially* on $M = S^{2n-1} \subset \mathbb{R}^n \oplus \mathbb{R}^n$ by $g \cdot (x, y) = (gx, gy)$, and set

$$\tau : S^{n-1} \to O(n), \qquad \tau(p)v = 2\langle p, v\rangle p - v, \qquad \text{i.e.} \qquad \tau(p) = 2pp^{\mathsf{T}} - I.$$

property	proof
$\tau(p)^{\mathsf{T}} = \tau(p)$, $\tau(p)^2 = I$	$(2pp^{\mathsf{T}} - I)^2 = 4pp^{\mathsf{T}}pp^{\mathsf{T}} - 4pp^{\mathsf{T}} + I = I$, using $p^{\mathsf{T}}p = 1$
$\det \tau(p) = (-1)^{n-1}$	$+1$ on $\mathbb{R}p$, -1 on $p^{\perp}$; so τ lands in $SO(n)$ iff n odd
$\tau(gp) = g\tau(p)g^{-1}$	$2(gp)(gp)^{\mathsf{T}} - I = g(2pp^{\mathsf{T}} - I)g^{\mathsf{T}}$

τ is the characteristic (clutching) map of TS^n. On the overlap $U_1 \cap U_2 = \{x \neq 0 \neq y\}$ put

$$\boxed{a(x, y) = \tau(\hat{x})\,\tau(\hat{y}).}$$

Conjugation-equivariance is again one line, $a(gx, gy) = g\tau(\hat{x})g^{-1}g\tau(\hat{y})g^{-1} = g\,a(x, y)\,g^{-1}$, and $\det a = 1$, so a takes values in $SO(n)$.

Punchline to state: II.1 and II.4 are the same theorem applied twice. Two one-line verifications, two exotic spheres.

II.5 The resulting $\star$-diagram, explicitly [2 min]

Write S^{2n-1} as the union of two solid tori,

$$S^{2n-1} = \big(D^n \times S^{n-1}\big) \;\cup\; \big(S^{n-1} \times D^n\big),$$

glued along the equatorial torus $S^{n-1} \times S^{n-1} = \{|x| = |y| = 1/\sqrt{2}\}$. Both pieces are $O(n)$-invariant, and a restricted to the equator is the clutching function.

The bundle. $P_a \to S^{2n-1}$ is the principal $SO(n)$-bundle obtained by gluing $\big(D^n \times S^{n-1}\big) \times SO(n)$ to $\big(S^{n-1} \times D^n\big) \times SO(n)$ along the equator via a. (Speranca identifies P_a with a pullback of the frame bundle $O(n+1) \to S^n$ along an explicit equivariant map; state this only if asked, and check the formula first.)

The twisted sphere. By Theorem 8 the $\star$-quotient is

$$P_a/\!\star \;\cong\; \big(D^n \times S^{n-1}\big) \;\cup_{\hat{a}}\; \big(S^{n-1} \times D^n\big),$$

where $\hat{a}$ is the reentrance diffeomorphism of the equatorial torus. It has a closed form: writing $s = \langle x, y\rangle$ for $x, y \in S^{n-1}$, and using $\tau(y)x = 2sy - x$, $\tau(x)y = 2sx - y$, $\tau(p)p = p$,

$$\boxed{\hat{a}(x, y) = \big(\tau(x)\tau(y)x,\ \tau(x)\tau(y)y\big) = \Big(\big(4s^2 - 1\big)\,x - 2s\,y,\quad 2s\,x - y\Big).} \tag{3}$$

Sanity checks to do aloud: $s = 1$ gives the identity; $s = 0$ gives the antipodal map $(x, y) \mapsto (-x, -y)$; and both components have unit norm since $\tau(x)\tau(y) \in SO(n)$.

Summary. The $\star$-diagram is

$$S^{2n-1} \;\longleftarrow\; P_a \;\longrightarrow\; \partial P^{2n}(A_2),$$

and for n odd the right-hand term is the Kervaire sphere Σ^{4m+1}, $n = 2m + 1$. In words:

> *S^{2n-1} and the Kervaire sphere are glued from the same two solid tori; they differ only in that the second uses the regluing (3) of the equatorial torus in place of the identity.*

This sentence is the one that Part V will act on.

Part III. Geometry [7 min]

Theorem 16 ([K]). *Given a G-invariant metric on M there is a G-invariant metric on $M^\star$ with $M^\star/G$* isometric *to M/G. Corresponding isotropy groups are isomorphic and slice representations equivalent — after a choice of point in the fibre of P, hence up to inner conjugation.*

State the caveat; it is genuinely there. Slogan for the board:

> M and $M^\star$ have the same orbit space, the same isotropy data, and possibly different smooth structures.

Mechanism. Both quotient maps out of P are Riemannian submersions for a suitable $G \times G$-invariant metric, so O'Neill and Cheeger deformations transport curvature conditions across the diagram: $\sec \geq 0$ on Σ^7; invariant positive Ricci curvature on the $U(k)$- and $Sp(k)$-quotients; almost non-negative curvature via Searle–Wilhelm.

Picture. Orbits as circles; $\hat{a}(x) = a(x) \cdot x$ slides each point along its own orbit by an amount varying with x; it is the identity on the fixed-point set, and the twisting concentrates where the action is free.

Caveat needed in Part IV. The biaxial $O(n)$-action on S^{2n-1} has *no* fixed points. For the connected-sum and Ricci machinery one replaces $O(n)$ by $U(k)$ acting biaxially on S^{4k+1}, $n = 2k+1$.

Part IV. The symplectic question [14 min]

IV.1 Basic definitions [3 min]

Definition 17 (cotangent bundle). $\pi : T^*M \to M$; tautological 1-form $\lambda_\eta(w) = \eta\big(d\pi_\eta w\big)$; canonical symplectic form $\omega = -\mathrm{d}\lambda$. In canonical coordinates $\lambda = p_i \, \mathrm{d}q^i$.

Definition 18 (Lagrangian). $L^n \subset (X^{2n}, \omega)$ with $\omega|_L = 0$. The zero section $0_M \subset T^*M$ is Lagrangian. L is *exact* if $\lambda|_L$ is exact.

Definition 19 (cotangent lift). For a diffeomorphism Φ of M, $\Phi_*(\xi_x) := \xi_x \circ (d\Phi_x)^{-1} \in T^*_{\Phi(x)}M$.

Lemma 20 (cotangent lifts preserve λ). $\Phi_*^*\lambda = \lambda$; *hence Φ_* is an exact symplectomorphism.*

Proof. $\pi \circ \Phi_* = \Phi \circ \pi$, so $d\pi \circ d\Phi_* = d\Phi \circ d\pi$. Then

$$(\Phi_*^*\lambda)_\eta(w) = (\Phi_*\eta)\big(d\pi \, d\Phi_* \, w\big) = (\Phi_*\eta)\big(d\Phi \, d\pi \, w\big) = \eta \circ (d\Phi)^{-1} \circ d\Phi\big(d\pi \, w\big) = \lambda_\eta(w).$$

□

Lemma 21 (moment map on T^*M). *The lifted G-action on T^*M preserves λ. Setting $\mu_X := \iota_{\zeta_X}\lambda$, Cartan's formula gives $\mathrm{d}\mu_X = \iota_{\zeta_X}\omega$; equivariance of π gives the explicit value*

$$\big\langle \mu(\eta_x), X \big\rangle = \eta_x\big(\zeta_X(x)\big),$$

and μ is Ad^*-equivariant. Note $\mu^{-1}(0)$ is the conormal bundle of the orbit foliation.

Proof. $\mathrm{d}\mu_X = \mathrm{d}\iota_{\zeta_X}\lambda = \mathcal{L}_{\zeta_X}\lambda - \iota_{\zeta_X}\mathrm{d}\lambda = 0 + \iota_{\zeta_X}\omega$ by Lemma 20 applied to $\Phi = L_g$; and $\mu_X(\eta) = \lambda_\eta(\zeta_X) = \eta(d\pi \, \zeta_X) = \eta(\zeta_X(x))$. Equivariance follows from (1). □

IV.2 Lifting the $\star$-diagram, and the problem [3 min]

By Theorem 10(2) with $E = T^*M$ there is a $\star$-diagram $T^*M \leftarrow \pi^*P \rightarrow (T^*M)^\star$. But this is only the pullback of the original diagram, so by itself it is empty. The real question is

$$\boxed{\text{Is} \quad (T^*M)^\star \;\cong\; T^*\big(M^\star\big) \;?}$$

Both are assembled from the same pieces T^*U_i, and both use gluings covering the same base map Φ_{ij} of (2). They differ in the fibre:

$$(\Phi_{ij})_*\xi = \xi \circ \big(d\Phi_{ij}\big)^{-1} \qquad \text{versus} \qquad \Psi_{ij}\xi = \xi \circ \big(dL_{\varphi_{ij}(x)}\big)^{-1}.$$

The first inverts the derivative of the whole reentrance map; the second inverts the derivative of the *frozen* group element. Equivalently, in the invariant description,

$$(T^*M)^\star = \mathrm{Ann}(V_\bullet)\big/\star, \qquad T^*(M^\star) = \mathrm{Ann}(V_\star)\big/\star,$$

two different subbundles of T^*P quotiented by the same group. (Here $\mathrm{Ann}(V_\bullet) = \mu_\bullet^{-1}(0)$.)

IV.3 The theorem [4 min]

State it with *no indices*: it is a purely local statement about a single map and never uses the cocycle condition. Say once that the regluing maps of $M^\star$ are of this form, with $a = \varphi_{ji}$.

Standing data. $U \subseteq M$ open and G-invariant; $a : U \to G$ with $a(g \cdot x) = g\,a(x)\,g^{-1}$; $\tilde\theta := a^{-1}\mathrm{d}a$; and

$$\Phi(x) = a(x) \cdot x, \qquad \Psi(\xi) = \xi \circ \big(dL_{a(\pi\xi)}\big)^{-1}, \qquad A_x(w) := \zeta_{\tilde\theta_x(w)}(x).$$

Theorem 22 ([N])**.**

$$\big(\Psi^*\lambda\big)_\xi(w) \;=\; \xi\Big(\big(\mathrm{Id} + A_{\pi\xi}\big)\,d\pi\,w\Big),$$

and the following are equivalent:

$$\Psi \text{ is symplectic} \iff \Psi^*\lambda = \lambda \iff A \equiv 0 \iff \tilde\theta_x(T_xM) \subseteq \mathfrak{g}_x \quad \forall x.$$

In particular, wherever the action is locally free: Ψ *is symplectic if and only if* a *is locally constant, i.e.* ***if and only if the $\star$-construction is trivial.***

Proof. Ψ covers Φ, so $d\pi \circ d\Psi = d\Phi \circ d\pi$. Writing Φ as the composite $x \mapsto (a(x), x) \mapsto a(x)\cdot x$ and differentiating the action map $\mathcal{A}(g, x) = g\cdot x$, whose derivative is $d\mathcal{A}(\dot g, \dot x) = dL_{g_0}\dot x + \zeta_{\theta_R(\dot g)}(g_0\cdot x_0)$, then using $\theta_R = \mathrm{Ad}_a\,\tilde\theta$ and (1):

$$d\Phi_x = dL_{a(x)} \circ \big(\mathrm{Id} + A_x\big). \tag{4}$$

Therefore

$$(\Psi^*\lambda)_\xi(w) = (\Psi\xi)\big(d\pi\, d\Psi\, w\big) = (\Psi\xi)\big(d\Phi\, d\pi\, w\big) = \xi\big((\mathrm{Id} + A)\, d\pi\, w\big).$$

So $\beta := \Psi^*\lambda - \lambda$ satisfies $\beta_\xi(w) = \xi\big(A\, d\pi w\big)$, which vanishes identically iff $A \equiv 0$. Finally, in a chart (t, η) one has $\beta = \sum_{i,j} \eta_i A^i{}_j(t)\,\mathrm{d}t^j$, so at $\eta = 0$, $\mathrm{d}\beta|_0 = \sum_{i,j} A^i{}_j\,\mathrm{d}\eta_i \wedge \mathrm{d}t^j$; hence $\mathrm{d}\beta = 0 \Rightarrow A = 0$. □

Corollary 23 (no non-exact symplectic case)**.** *The following are equivalent:* (i) $\Psi^*\omega = \omega$*;* (ii) $\Psi^*\lambda = \lambda$*;* (iii) $\beta = 0$*;* (iv) $A \equiv 0$*. In particular* Ψ *cannot be a symplectomorphism which fails to be exact.*

Corollary 24 (no twisted symplectic form helps)**.** *For a closed $B \in \Omega^2(M)$, the* twisted cotangent bundle *is $\big(T^*M,\ \omega + \pi^*B\big)$. If $\Psi^*\big(\omega + \pi^*B\big) = \omega$ for some closed $B \in \Omega^2(M)$, then $B = 0$ and $A \equiv 0$.*

Proof. Corollary 23 is Theorem 22 restated, since $\Psi^*\omega = \omega - \mathrm{d}\beta$. For Corollary 24, note $\Psi^*\pi^*B = (\pi \circ \Psi)^*B = (\Phi \circ \pi)^*B = \pi^*(\Phi^*B)$, so the hypothesis reads $\mathrm{d}\beta = \pi^*(\Phi^*B)$. Evaluate both sides at a point 0_x of the zero section on pairs of vectors with vanishing fibre component: the right-hand side gives $(\Phi^*B)(w_1, w_2)$, the left-hand side gives 0. Hence $\Phi^*B = 0$, so $B = 0$ since Φ is a diffeomorphism; then $\mathrm{d}\beta = 0$ and Theorem 22 gives $A \equiv 0$. □

Remark 25 (free consistency check)**.** Φ is a diffeomorphism, so $\mathrm{Id} + A_x$ is invertible everywhere. Use this to test any computation of A.

IV.4 The closing computation [4 min]

Take $M = S^{2n-1}$, $G = SO(n)$ biaxial, $a(x, y) = \tau(\hat{x})\tau(\hat{y})$ as in II.4. Write $u \wedge v := uv^{\mathsf{T}} - vu^{\mathsf{T}}$, so that $(u \wedge v)w = \langle v, w\rangle u - \langle u, w\rangle v$.

Step 1. For $|p| = 1$ and $\delta p \perp p$,

$$\tau(p)^{-1}\mathrm{d}\tau_p(\delta p) = (2pp^{\mathsf{T}} - I) \cdot 2\big(\delta p\, p^{\mathsf{T}} + p\, \delta p^{\mathsf{T}}\big) = 2\, p \wedge \delta p.$$

Step 2. Using $(bc)^{-1}\mathrm{d}(bc) = \mathrm{Ad}_{c^{-1}}(b^{-1}\mathrm{d}b) + c^{-1}\mathrm{d}c$, $\mathrm{Ad}_{\tau(q)}(u \wedge v) = (\tau(q)u) \wedge (\tau(q)v)$, and $R := \tau(\hat{y}) = \tau(\hat{y})^{-1}$,

$$\tilde{\theta} = 2\,(R\hat{x}) \wedge (R\,\mathrm{d}\hat{x}) \ +\ 2\,\hat{y} \wedge \mathrm{d}\hat{y}.$$

Step 3. At $z_0 = \frac{1}{\sqrt{2}}(p, q)$ with $p \perp q$, take $w = (q, 0)$. Then $\mathrm{d}\hat{x} = \sqrt{2}\, q$, $\mathrm{d}\hat{y} = 0$, $Rp = -p$, $Rq = q$, so

$$\tilde{\theta} = 2\sqrt{2}\,(Rp) \wedge (Rq) = -2\sqrt{2}\, p \wedge q.$$

Step 4. Since $(p \wedge q)p = -q$ and $(p \wedge q)q = p$,

$$A(w) = \tfrac{1}{\sqrt{2}}\big(\tilde{\theta}\, p,\ \tilde{\theta}\, q\big) = \big(2q,\ -2p\big).$$

Step 5. With $\xi = (q, 0)^\flat$,

$$\beta_\xi(w) = \xi\big(Aw\big) = \big\langle (q, 0), (2q, -2p)\big\rangle = 2 \ \neq\ 0.$$

Corollary 26. *$A \neq 0$. Hence, by Theorem 22, the $\star$-regluing of T^*S^{2n-1} induced by $a(x, y) = \tau(\hat{x})\tau(\hat{y})$ is not a symplectomorphism of (T^*S^{2n-1}, ω); and by Corollary 24 it is not a symplectomorphism onto $(T^*S^{2n-1}, \omega + \pi^*B)$ for any closed $B \in \Omega^2(S^{2n-1})$ either.*

Additional structure, if time permits. $\det(\mathrm{Id} + A) \equiv 1$ and $\mathrm{spec}(\mathrm{Id} + A) = \{1, 1, 1\} \cup \{e^{\pm 2i\psi}\}^{n-2}$, where $\psi = \angle(p, q)$, the eigenvalue 1 carrying a Jordan block of size 2. Two consequences, stated precisely: $\det d\Phi = \det dL_a \cdot \det(\mathrm{Id} + A) = 1$, so Φ preserves the Riemannian volume of the round metric; and $\mathrm{Id} + A$ is not semisimple, hence not conjugate to an element of $O(2n - 1)$, so Φ is not an isometry.

The quaternionic case runs identically. For $a = \hat{x}\overline{\hat{y}}$ on S^7,

$$\tilde{\theta} = \hat{y}\,\big(\overline{\hat{x}}\,\mathrm{d}\hat{x}\big)\,\overline{\hat{y}} - \mathrm{d}\hat{y}\,\overline{\hat{y}},$$

and at $\frac{1}{\sqrt{2}}(j, i)$ with $w = (\frac{1}{\sqrt{2}}, 0)$ one gets $A(w) = (0, -\sqrt{2}\, k)$ and $\beta = \sqrt{2}$.

Part V. Cotangent bundles of homotopy spheres [12 min]

V.1 Why cotangent bundles were the question

Part II produced, from an explicit conjugation-equivariant map, two manifolds

$$S^{4m+1} \qquad \text{and} \qquad \Sigma^{4m+1} = \partial P^{4m+2}(A_2)$$

which are homeomorphic, usually not diffeomorphic, and are glued from *the same two solid tori* — the second using the regluing (3) in place of the identity. The natural symplectic question is whether that difference survives passage to cotangent bundles. It does not merely rhyme with a known open problem; it is that problem.

Definition 27 (nearby Lagrangian)**.** A closed exact Lagrangian $L \subset T^*Q$.

Conjecture (Arnold). Every closed exact Lagrangian $L \subset T^*Q$ is Hamiltonian isotopic to the zero section; in particular L is diffeomorphic to Q.

Two remarks that make the conjecture bite here.

(a) For L a homotopy sphere, exactness is automatic: $\lambda|_L$ is closed and $H^1(L;\mathbb{R}) = 0$.

(b) If $T^*\Sigma$ and T^*S^{4m+1} were symplectomorphic, then the zero section of the first would give a closed exact Lagrangian in the second, diffeomorphic to Σ. So a symplectomorphism of cotangent bundles of exotic spheres would *disprove* the conjecture.

V.2 The obstruction landscape, and why bP is invisible to it

result	statement
Abouzaid–Kragh	For closed Q, the projection $L \to Q$ is a simple homotopy equivalence. Hence a nearby Lagrangian in T^*S^N is a homotopy N-sphere.
Abouzaid (*Annals* 2012)	A homotopy sphere embedding as a Lagrangian in T^*S^{4k+1} must bound a compact parallelisable manifold.
Ekholm–Kragh–Smith (2016)	For $k > 2$, $T^*\Sigma \cong T^*\Sigma'$ symplectically implies $[\Sigma] = [\pm\Sigma']$ in Θ_{2k-1}/bP_{2k}.
Álvarez-Gavela (2026)	If a homotopy N-sphere L embeds Lagrangianly in T^*M for a homotopy N-sphere M, then $[L] - [M]$ is a multiple of η in Θ_N/bP_{N+1}; in dimension 8, $L \cong S^8$.

Every one of these invariants is computed in Θ_N/bP_{N+1}.

The Kervaire sphere *generates* bP_{4m+2}. So $[\Sigma^{4m+1}] = 0$ in that quotient, and every known obstruction vanishes on it identically. This is structural, not a gap somebody forgot to fill: the methods are insensitive to bP by construction.

V.3 Dimension nine: a single question

Put this on the board.

Proposition 28 (assembling the above)**.** *A nearby Lagrangian in T^*S^9 is a homotopy 9-sphere bounding a parallelisable manifold. Now $\Theta_9 \cong (\mathbb{Z}/2)^3$ has eight elements, of which exactly two bound parallelisable manifolds, so $bP_{10} = \{S^9, \Sigma^9\}$. Hence the nearby Lagrangian conjecture for T^*S^9 is equivalent to the single question:*

does the Kervaire 9-sphere embed as an exact Lagrangian in T^*S^9?

V.4 What Theorem 22 does, and does not, say about this

The $\star$-diagram of II.5 offers an obvious strategy: Σ^{4m+1} is S^{4m+1} reglued along (3), so try to perform the same regluing on cotangent bundles and obtain a symplectomorphism $T^*\Sigma \cong T^*S^{4m+1}$.

What is proved. Theorem 22 together with Corollary 26 closes that route completely. The canonical lift of the $\star$-construction to T^*M is symplectic precisely when it is trivial, and for $a(x,y) = \tau(\hat{x})\tau(\hat{y})$ it is not trivial, since $A \neq 0$ with an explicit witness. Corollary 24 also removes the fallback of replacing ω by a twisted form $\omega + \pi^*B$. As a by-product this subsumes the natural first attempt — taking the cocycle to be a collective Hamiltonian $t = \exp(\mathrm{d}\chi \circ \mu)$ for an Ad^*-invariant χ — since such a t has $\hat{t}$ Hamiltonian isotopic to the identity and hence produces no twist at all.

What is not proved. Nothing about Arnold. Theorem 22 concerns one specific map, Ψ. Its failure to be symplectic says the $\star$-machine does not lift; it says nothing about whether some *other* symplectomorphism $T^*\Sigma \cong T^*S^{4m+1}$ exists. The correct reformulation is this. The cotangent lift $(\hat{a})_*$ of (3) is automatically an exact symplectomorphism, by Lemma 20. Hence $T^*\Sigma$ and T^*S^{2n-1} are both assembled from $T^*\big(D^n \times S^{n-1}\big)$ and $T^*\big(S^{n-1} \times D^n\big)$, glued by $(\hat{a})_*$ and by the identity respectively. So everything reduces to a π_0-question:

> *Is $(\hat{a})_*$ isotopic to the identity through exact symplectomorphisms of the collar, with the appropriate behaviour at infinity?*

If yes, the conjecture fails. If no, nothing follows. Our contribution is that (3) is completely explicit and $O(n)$-equivariant, and that (4) gives its derivative in closed form.

V.5 Approaches to a counterexample [4 min]

Remark 29 (a precision that widens the search)**.** The statements
(i) $T^*\Sigma$ and T^*S^N are symplectomorphic;
(ii) Σ embeds as a closed exact Lagrangian in T^*S^N;
are not equivalent: (i) implies (ii), by taking the image of the zero section, but not conversely. A counterexample to the conjecture requires only (ii). Route A below pursues (i); Routes B and C pursue (ii) directly.

Route A: the symplectic mapping class group. By Lemma 20, $(\hat{a})_*$ is an exact symplectomorphism, and T^*S^{2n-1} and $T^*\Sigma$ are assembled from $T^*(D^n \times S^{n-1})$ and $T^*(S^{n-1} \times D^n)$ glued by the identity and by $(\hat{a})_*$ respectively. Everything is therefore a π_0 question.

Question 1. Is $(\hat{a})_*$ isotopic to the identity through exact symplectomorphisms of the collar, with prescribed behaviour at infinity?

Question 2. Describe the kernel of the cotangent-lift homomorphism $\pi_0\mathrm{Diff}(N) \to \pi_0\,\mathrm{Symp}^{\mathrm{ex}}(T^*N)$ for $N = S^{n-1} \times S^{n-1}$: which mapping classes become symplectically trivial?

Question 3. What is the class of $\hat{a}$ in $\pi_0\mathrm{Diff}(S^{n-1} \times S^{n-1})$, and where does the associated twisted sphere sit in the Gromoll filtration of $\pi_0\mathrm{Diff}_\partial(D^{4m})$? A twist supported in a small disc and a twist spread over the whole manifold behave differently under symplectic isotopy.

Route B: direct Lagrangian embedding.

Question 4. Does Σ^{4m+1} admit a closed exact Lagrangian embedding into T^*S^{4m+1}? Candidate techniques: begin with an exact Lagrangian immersion and cancel double points (Ekholm–Eliashberg–Murphy–Smith); or perform Lagrangian surgery on a configuration of spheres. The cut-and-paste of Ekholm–Kragh–Smith already produces exact Lagrangian spheres inside plumbings of cotangent bundles; the question is whether the ambient manifold can be taken to be T^*S^N.

Question 5. Equivariant version. Σ^{4m+1} is cohomogeneity one for $SO(2)SO(n)$ and T^*S^{4m+1} carries the corresponding Hamiltonian action. Does an *invariant* exact Lagrangian exist? This would reduce the problem to an ODE on the orbit space.

Route C: Weinstein homotopy and stabilisation. $DT^*\Sigma$ and DT^*S^N are Weinstein domains with identical homotopy-theoretic data. They are not flexible, so Eliashberg–Murphy does not apply; but Courte has shown that exotic smooth phenomena can become symplectically invisible in the flexible range.

Question 6. Are $DT^*\Sigma$ and DT^*S^N Weinstein homotopic? Are the stabilisations $DT^*\Sigma \times \mathbb{C}$ and $DT^*S^N \times \mathbb{C}$ symplectomorphic?

Question 7. A necessary condition for (i) of Remark 29: is the contact manifold $ST^*\Sigma$ contactomorphic to ST^*S^N?

Route D: an invariant sensitive to bP. This is the obstruction side, and structurally the sharpest question. One has $bP_{N+1} = \ker\big(\Theta_N \to \operatorname{coker} J\big)$, and every published obstruction factors through framed bordism, hence through $\operatorname{coker} J$, hence vanishes on bP. Detecting bP_{4m+2} requires the Arf invariant of a quadratic refinement of the intersection form on a *parallelisable coboundary* — information about a coboundary, not about a framing.

Question 8. Is there a Floer-theoretic or microlocal-sheaf-theoretic invariant of a nearby Lagrangian which remembers a parallelisable coboundary together with a quadratic refinement of its intersection form — a "Floer-theoretic Kervaire invariant"?

Question 9. Weaker, and perhaps more accessible: does a nearby Lagrangian $L \subset T^*S^N$ bound a parallelisable Weinstein subdomain of T^*S^N compatibly with the Liouville structure? If so, the Arf invariant of that coboundary is a candidate obstruction.

Route E: structural.

Question 10. Does the $\star$-diagram implement a symplectic Morita equivalence between the Poisson manifolds $\mathfrak{g}^* \times M$ and $\mathfrak{g}^* \times M^\star$, with T^*P as bimodule?

Where to test. $\Theta_5 = 0$, and by Theorem 15 the Kervaire spheres in dimensions $13, 29, 61, 125$ are standard. The first genuinely exotic case is Σ^9, then $17, 21, 25, 33, \ldots$. Dimension 9 is where the conjecture reduces to a single question, so every construction should be tested there first.

V.6 Closing sentence

Do *not* claim to disprove Arnold. Close with:

> I have an explicit, equivariant model of the unique surviving candidate; I have computed in closed form the endomorphism measuring the failure of the construction to be symplectic, and proved that this failure persists after replacing the canonical symplectic form by any twisted form; and I would like to know whether the underlying π_0-question has an answer. That is what I am asking this room.

Appendix A. Pocket card

- $\zeta_{\mathrm{Ad}_g X}(g \cdot x) = dL_g \zeta_X(x)$
- $\mu_X = \iota_{\zeta_X}\lambda, \quad \mathrm{d}\mu_X = \iota_{\zeta_X}\omega$
- $\mathrm{d}\theta_L + \frac{1}{2}[\theta_L, \theta_L] = 0, \quad \theta_R = \mathrm{Ad}_g\, \theta_L$
- $(bc)^{-1}\mathrm{d}(bc) = \mathrm{Ad}_{c^{-1}}(b^{-1}\mathrm{d}b) + c^{-1}\mathrm{d}c$
- $(u \wedge v)w = \langle v, w\rangle u - \langle u, w\rangle v$
- $\langle a \wedge b, c \wedge d\rangle = \langle a, d\rangle\langle b, c\rangle - \langle a, c\rangle\langle b, d\rangle$
- $\tau(p) = 2pp^{\mathsf{T}} - I;\ \ \tau^2 = I;\ \ \det\tau = (-1)^{n-1};\ \ \tau^{-1}\mathrm{d}\tau = 2\,p \wedge \delta p$
- $\hat{a}(x, y) = \big((4s^2 - 1)x - 2sy,\ 2sx - y\big), \ \ s = \langle x, y\rangle$
- $Q = \left(\begin{smallmatrix} \chi(S^n) & 1 \\ 1 & \chi(S^n) \end{smallmatrix}\right);\ \ \det Q = -1$ (n odd), 3 (n even)
- $\Theta_9 \cong (\mathbb{Z}/2)^3,\ \ bP_{10} \cong \mathbb{Z}/2$
- Kervaire sphere standard only in dimensions $5, 13, 29, 61, 125$

Appendix B. Timing

part	minutes	cut first?
0. The question	4	no
I. The engine	14	no
II. Examples (incl. plumbing)	18	II.1 fixed-point data
III. Geometry	7	compress to 4
IV. The symplectic question	14	no — this is the point
V. Cotangent bundles	12	V.5 Questions 4–5
total	69	trim to ∼55 by compressing III and cutting Q4–Q5